REFERENTIAL AND SPATIAL EVOLUTIONS IN NONLINEAR ELASTICITY

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REFERENTIAL AND SPATIAL EVOLUTIONS IN NONLINEAR ELASTICITY

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To my parents, Nassim and Kamal,

to my aunt Mounounou,

and to my siblings, Nouha and Raouph.

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SUMMARY

In this PhD thesis, we propose a theoretical framework for studying referential and spatial evolutions in nonlinear elasticity. We use the referential evolution—considering an evolving reference configuration—to formulate a geometric theory of anelasticity. Indeed, an anelasticity source (such as temperature, defects, or growth) can manifest itself such that the body would fail to find a relaxed state in the Euclidean physical space. However, a reference configuration should by essence be stress-free so that one can properly quantify the strain with respect to it—and the stress by means of a constitutive equation. Identifying the reference configuration with an abstract manifold—material manifold—allows for a rational construction of such a stress-free state which can further accommodate the evolution of the source of anelasticity by allowing the material manifold to have an evolving geometry. In this work, we formulate a general geometric theory of anelasticity for three-dimensional bodies that we apply to the particular case of thermoelasticity; and we also formulate a general theory of anelastic shells that we apply to the particular case of morphoelastic shells, i.e., those subject to growth and remodeling. In the context of anelasticity, as well as in nonlinear elasticity, most exact solutions are obtained by assuming some restrictive class of symmetry for the solution. We propose a theory of small-on-large anelasticity, that is analogous to the small-on-large theory of Green et al. in classical elasticity. It can be used to find exact solutions for non-symmetric distributions of anelasticity sources that are small perturbations of symmetric ones. Finally, motivated by gaining further insights on the theory of nonlinear elasticity as well as the case of a continuum deforming in an evolving ambient space, we formulate a theory of nonlinear elasticity where the geometry of the ambient space is time-dependent.

CHAPTER I

INTRODUCTION

The theory of elasticity is concerned with the study of elastic materials, that is, materials that restore their initial shape after deformation. This theory has long been limited to the linear approximation assuming infinitesimal strain, i.e., small deformations. In linear elasticity, the stress-strain relationship is assumed to be linear in the form of the so-called Hooke's law. In many classical engineering applications, these assumptions are reasonable and are valid to some extent in a wide range of applications. However, beyond the infinitesimal regime, finite elastic deformations require a less restrictive framework, hence the theory of nonlinear elasticity. It can be used to model for example rubber-like materials [57, 58] and soft biological tissues [15]. Note that nonlinearity in elasticity concern both the material nonlinearity, i.e., a nonlinear constitutive model, and the geometrical nonlinearity, i.e., taking a nonlinearized deformation measure.

In elasticity, the reference configuration represents a set of material particles arranged in such a way that they form a stress-free continuum that deforms within a continuous ambient space such as the physical space. Note that the continuum model is a representation of the physical reality that is merely a physical analogue of the notion of a differentiable manifold. Hence, the reference configuration and the ambient space where it deforms can be both be identified as differentiable manifolds. A deformation of the body in the ambient space is also naturally represented by a mapping—more or less smooth depending on the nature of the deformation—between the reference configuration and the ambient space. Endowing these manifold with metrics, one can conveniently measure distances, and by using the notions of push forward and pull back, one can naturally define measures of strain to quantify the deformation.

Beyond pure elasticity, taking the reference configuration as an abstract manifold allows for the construction of a stress-free material (referential) configuration even when the body cannot relax in the physical space leading to residual stresses, i.e., anelasticity. Anelastic effects can exist in the body such that it has a configuration that cannot be relaxed in the Euclidean space, i.e., one cannot construct a Euclidean referential stress-free configuration. However, such a configuration can be realized as an abstract manifold. Such a construction is done by endowing the material manifold with a metric that accounts for the anelastic-distortions such that the material manifold remains stress-free. In an anelastic deformation, any measure of strain has a non-elastic component. This means that a non-vanishing strain does not necessarily correspond to a non-vanishing (conjugate) stress; only the elastic part of strain enters the constitutive equations. The remaining part of strain is called pre-strain or *eigenstrain* as coined by Mura [52]. Examples of anelasticity sources include defects [91, 90, 88], non-uniform temperature distributions [81, 62, 70], bulk growth [68, 20, 5, 89, 72], accretion (surface growth) [55, 78], and swelling [64, 65, 66]. By constructing the stress-free reference configuration as an abstract manifold, anelasticity is readily reduced to nonlinear elasticity on an abstract material manifold with a non-trivial material metric. Besides, due to the dynamical nature of the sources of anelasticity (temperature, defect, growth, *etc*), the material metric ought to be time-dependent to ensure that the material manifold remains stress-free at all times—hence *referential evolution*.

In nonlinear elasticity, and more so in anelasticity, the governing equations present such a level of complexity than solutions are almost exclusively found by using symmetry-based inversed methods. Hence, the vast majority of the classes of known solutions are highly symmetric. One way for extending the class of problems amenable to exact solutions in anelasticity is to study those distributions that are perturbations of the highly symmetric ones. This is what we call small-on-large anelasticity, which is a material analogue of the small-on-large theory of Green et al. [28] (further discussion and several applications of this theory can be found in [29, 84]). Given a distribution of some source of anelasticity with a known exact solution, we perturb the distribution and solve for the induced small elastic deformations. This is achieved by linearizing the governing equations about the known solution with respect to the perturbation. Even in the case when one fails to find exact solutions in this framework, the linearized governing equations are much easier to solve numerically.

In the geometric field theory of elasticity, the spatial metric is introduced as a fixed background geometry. Likewise, in the classical theory of nonlinear elasticity, this background metric is a given geometric object with no dynamics. Motivated by the hope of gaining a deeper understanding of the structure of the classical theory, we relax this assumption and study the case of *spatial evolution* which consists in considering an ambient space with an evolving geometry.[1] Our motivation comes also from possible applications of this theory involving the analysis of elastic bodies constrained to move on curved, dynamical surfaces. One example of such a situation is the case of a shell constrained to deform on a sphere of time-dependent radius such as is the case in the formation of coated vesicles or the biogenesis of multivesicular bodies. We formulate the spatial evolution by taking the material metric as fixed, but we consider an evolving spatial metric via a time-dependent embedding of the ambient space in a larger space with a fixed background metric.

This work is arranged as follows. In Chapter 2, we present a geometric theory of anelasticity of three-dimensional bodies and illustrate the capabilities of the theory

[1] The generalization of a theory obtained by relaxing certain standard assumptions (in this case, the staticity of $\boldsymbol{g}_t$), commonly results in a deeper understanding of the original theory. Examples of this include the geometric notions of stress and traction obtained by allowing the spatial metric to be non-Euclidean.

by solving examples from thermoelasticity [70]. In Chapter 3, we present a geometric theory of anelasticity of shells and illustrate the theory by looking at the particular case of morphoelastic shells [72], i.e., shells that are subject to bulk growth and remodeling. In Chapter 4, we present our theory of small-on-large anelasticity which can be used to find exact solutions of some non-symmetric eigenstrain distributions and apply it to the case of screw dislocations where we find an exact solution for a non-symmetric distribution [71]. In Chapter 5, we present a theory of nonlinear elasticity in a deforming ambient space [97]. Finally, in Chapter 6, we present our concluding remarks and briefly discuss future research directions along the lines of the presented work.

CHAPTER II

ANELASTICITY OF THREE-DIMENSIONAL BODIES

In this section, we formulate a geometric theory of three-dimensional anelasticity that can be used to predict the evolution of the residual stress fields in a nonlinear elastic body due to the presence of some source of anelasticity such as temperature, bulk growth, defects, *etc.* In this theory the material manifold (natural stress-free configuration of the body) is a Riemannian manifold with a time-dependent metric that depends explicitly on the anelastic eigenstrain distribution. The evolution of the geometry of the material manifold is governed by the evolution of the anelastic eigenstrain distribution such that the material manifold remains stress-free. As an example, we apply this theory to the case where anelasticity is due to the presence of a non-uniform temperature field and solve the problem of a spherical ball with a spherically-symmetric temperature distribution. Note that the results of this section have been previously reported in our published work [70].

2.1 Kinematics

We tersely review a few elements of the geometric formulation of the kinematics for three-dimensional nonlinear elasticity. For more details, see for example [50]. Let B be a three-dimensional body identified with a three-dimensional Riemannian manifold $\mathcal{B}$ endowed with a metric $\boldsymbol{G}$. The Riemannian manifold $(\mathcal{B}, \boldsymbol{G})$ represents the reference configuration of the body and will be referred to as the material manifold. Let the ambient space be represented by a three-dimensional Riemannian manifold $(\mathcal{S}, \boldsymbol{g})$.

Remark 2.1.1. Note that the material manifold need not be Riemannian. While one can construct the material metric for thermoelasticity directly as a Riemmanian metric, a detour via non-Riemmannian geometry is sometimes required to construct the stress-free material manifold, e.g., dislocations can be modeled by torsion [90, 63], and point defects by non-metricity [91]. However, only the underlying Riemannian metric is needed to calculate (residual) stresses. It is hence fair to assume in the following developments that the metric is Riemannian.

We adopt the standard convention to denote objects and indices by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$) and by lowercase characters in the spatial manifold $\mathcal{S}$ (e.g., $x \in \mathcal{S}$). Let $\{X^A\}$ and $\{x^a\}$ be local coordinate charts on $\mathcal{B}$ and $\mathcal{S}$, respectively. Also, let $\partial_A = \frac{\partial}{\partial X^A}$ and $\partial_a = \frac{\partial}{\partial x^a}$ denote the local coordinate bases corresponding to $\{X^A\}$ and $\{x^a\}$, respectively, and let $\{dX^A\}$ and $\{dx^a\}$ denote the corresponding dual bases. We also adopt Einstein's repeated index summation convention. We denote in the remainder of this section the Levi-Civita connections of the material manifold $(\mathcal{B}, \boldsymbol{G})$ and the ambient space $(\mathcal{S}, \boldsymbol{g})$ by ∇ and $\bar{\nabla}$, respectively.

A configuration of $\mathcal{B}$ is a smooth embedding $\varphi : \mathcal{B} \to \mathcal{S}$. We denote the set of all configurations of $\mathcal{B}$ by $\mathcal{C}$. A motion of $\mathcal{B}$ is a smooth curve $t \in \mathbb{R}^+ \to \varphi_t \in \mathcal{C}$ that assigns a spatial point $x = \varphi(X, t) = \varphi_t(X)$ at any time t to every material point X. For a fixed $X \in \mathcal{B}$ we write $\varphi_X(t) = \varphi(X, t)$. The material velocity of the motion is defined as the mapping

$$\boldsymbol{V} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S} \ \text{ such that } \ \boldsymbol{V}(X, t) = d_t\varphi_X \left[\frac{\partial}{\partial t}\right] \in T_{\varphi_X(t)}\mathcal{S} \,.$$

The spatial velocity is defined as the mapping

$$\boldsymbol{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S} \ \text{ such that } \ \boldsymbol{v}(x, t) = \boldsymbol{V}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S} \,.$$

The material acceleration is defined as the mapping

$$\boldsymbol{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S} \ \text{ such that } \ \boldsymbol{A}(X, t) = \bar{\nabla}_{\boldsymbol{V}(X,t)}\boldsymbol{V}(X, t) \in T_{\varphi(X)}\mathcal{S} \,.$$

In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc} V^b V^c$, where $\gamma^a{}_{bc}$ denote the Christoffel symbols of the connection $\bar{\nabla}$ in the local coordinate chart $\{x^a\}$, i.e., $\bar{\nabla}_{\partial_b}\partial_c = \gamma^a{}_{bc}\partial_a$. The spatial acceleration is defined as the mapping

$$\boldsymbol{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S} \quad \text{such that} \quad \boldsymbol{a}(x,t) = \boldsymbol{A}(\varphi_t^{-1}(x), t) \in T_x\mathcal{S} \,.$$

The deformation gradient $\boldsymbol{F}$ is defined as the tangent map of $\varphi_t : \mathcal{B} \to \mathcal{S}$, i.e.

$$\boldsymbol{F}(X,t) = d\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi(X)}\mathcal{S} \,.$$

The adjoint $\boldsymbol{F}^{\mathsf{T}}$ of $\boldsymbol{F}$ is defined by

$$\boldsymbol{F}^{\mathsf{T}}(X,t) : T_{\varphi(X)}\mathcal{S} \to T_X\mathcal{B} \,, \quad \forall (\boldsymbol{W}, \boldsymbol{w}) \in (T_X\mathcal{B} \times T_{\varphi_t(X)}\mathcal{S}) : \boldsymbol{g}(\boldsymbol{F}\boldsymbol{W}, \boldsymbol{w}) = \boldsymbol{G}(\boldsymbol{W}, \boldsymbol{F}^{\mathsf{T}}\boldsymbol{w}) \,.$$

In components, $(F^{\mathsf{T}})^A{}_a = g_{ab} F^b{}_B G^{AB}$. The Jacobian of the motion J relates the material and spatial Riemannian volume elements $dV(X, \boldsymbol{G})$ and $dv(x, \boldsymbol{g})$ by

$$dv(\varphi_t(X), \boldsymbol{g}) = J(X, \varphi_t, \boldsymbol{G}, \boldsymbol{g}) dV(X, \boldsymbol{G}) \,.$$

It can be shown that [50]

$$J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F} = e^{-\operatorname{tr}(\boldsymbol{\omega}(X,T))} \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}_0}} \det \boldsymbol{F} \,.$$

The right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C}(X,t) = \boldsymbol{F}^{\mathsf{T}}(X,t)\boldsymbol{F}(X,t) : T_X\mathcal{B} \to T_X\mathcal{B} \,.$$

In components, $C^A{}_B = G^{AK} F^a{}_K F^b{}_B g_{ab}$. We note that $\boldsymbol{C}^\flat$ agrees with the pull-back of the spatial metric $\boldsymbol{g}$ by φ_t , i.e., $\boldsymbol{C}^\flat = {\varphi_t}^*\boldsymbol{g}$, where $^\flat$ denotes the flat operator. The material strain tensor is defined as the difference between the pull back of the spatial metric and the material metric, i.e.

$$\boldsymbol{E} = \frac{1}{2} \left({\varphi_t}^*\boldsymbol{g} - \boldsymbol{G} \right) = \frac{1}{2} (\boldsymbol{C}^\flat - \boldsymbol{G}).$$

In components, $E_{AB} = \frac{1}{2} \left(C_{AB} - G_{AB} \right)$.

2.2 The material metric

In the theory of elasticity, one obtains the stress through a constitutive equation from some measure of strain. Such a measure of strain quantifies the deformation of the body in its current configuration with respect to a stress-free reference configuration. However, a stress-free configuration does not necessarily exist in the physical three-dimensional Euclidean space. In particular, in the presence of some source of anelasticity, the body could a configuration of that cannot be relaxed in the physical space and hence induces residual stresses. Following the pioneering works of Eckart [19] and Kondo [40], and assuming the existence of a hypothetic intermediate relaxed (stress-free) configuration, [6] and [41] independently introduced a multiplicative decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p$, into a plastic component $\boldsymbol{F}_p$ taking the reference configuration to the intermediate relaxed configuration and an elastic component $\boldsymbol{F}_e$ with respect to the intermediate configuration.[1] Note however that the decomposition $\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p$ is not unique and assumes a mathematically vague notion of an intermediate configuration. In this work, for the purpose of formulating a geometric theory of anelasticity, rather than considering an intermediate stress-free configuration, we endow the material manifold with a non-Euclidean metric such that the material configuration is a stress-free abstract manifold. Further, following the evolution of the source of anelasticity, the material metric is constructed as a dynamic variable so that the material configuration $(\mathcal{B}, \boldsymbol{G}(X,t))$ evolves accordingly as a stress-free reference configuration.

[1]See [48, 1] for an extensive review and a comprehensive bibliography on the decomposition of the deformation gradient in anelasticity and [73] for a historical perspective on the subject.

2.3 The governing equations of motion

Balance of mass. Let ρ and ϱ, respectively, denote the material and spatial mass densities. For any open set $\mathcal{U}$ in $\mathcal{B}$, conservation of mass can be written as

$$\int_{\varphi_t(\mathcal{U})} \varrho dv = \int_{\mathcal{U}} \rho dV \,. \tag{1}$$

where $dv = \sqrt{\det \boldsymbol{g}(x)} dx^1 \wedge dx^2 \wedge dx^3$ and $dV = \sqrt{\det \boldsymbol{G}(X,t)} dX^1 \wedge dX^2 \wedge dX^3$ are respectively the volume forms of the spatial and material manifolds. By applying the change of variable $X = \varphi_t^{-1}(x)$ to the right hand-side of (1) and by the arbitrariness of $\mathcal{U}$, we find that conservation of mass is equivalent to

$$\rho = J \varrho \,.$$

Note that unless there is any mass input/output involved such as is the case for growth, one has that $dM = \rho dV$ is constant. However, assuming a spatial rate of change of mass $s_m = s_m(x,t)$, the balance of mass reads

$$\frac{d}{dt}\left(\int_{\varphi_t(\mathcal{U})} \varrho dv\right) = \int_{\varphi_t(\mathcal{U})} s_m(x,t) dv \,.$$

Or equivalently

$$\frac{d}{dt}\left(\int_{\mathcal{U}} \rho dV\right) = \int_{\varphi_t(\mathcal{U})} S_m(x,t) dV \,,$$

where $S_m = J s_m$ is the material rate of change of mass. Therefore the balance of mass reads in local spatial form

$$\dot{\varrho} + \varrho \, \mathrm{div} \boldsymbol{v} = s_m \,,$$

where div denotes the spatial divergence operator. In local material form, the balance of mass reads

$$\dot{\rho} + \frac{1}{2} \rho \, \mathrm{tr} \dot{\boldsymbol{G}} = S_m \,. \tag{2}$$

Constitutive relations. We assume that the body is made of a hyperelastic material, so that the constitutive model is given by an energy function $\mathcal{W} = \tilde{\mathcal{W}}(X, \boldsymbol{C}^\flat, \boldsymbol{G})$ per unit undeformed volume, and define the following stress tensors:

$$\begin{aligned}
&\text{Second Piola-Kirchhoff stress tensor: } \boldsymbol{S} = 2\frac{\partial \mathcal{W}}{\partial \boldsymbol{C}^\flat}\,, \text{ in components,} \quad S^{AB} = 2\frac{\partial \mathcal{W}}{\partial C_{AB}}\,;\\
&\text{First Piola-Kirchhoff stress tensor: } \boldsymbol{P} = 2\boldsymbol{F}\frac{\partial \mathcal{W}}{\partial \boldsymbol{C}^\flat}\,, \text{ in components,} \quad P^{bB} = 2\frac{\partial \mathcal{W}}{\partial C_{AB}} F^b{}_A\,;\\
&\text{Cauchy stress tensor: } \boldsymbol{\sigma} = \frac{2}{\mathrm{J}}\boldsymbol{F}\frac{\partial \mathcal{W}}{\partial \boldsymbol{C}^\flat}\boldsymbol{F}^\mathsf{T}\,, \text{ in components,} \quad \sigma^{ab} = \frac{2}{\mathrm{J}}\frac{\partial \mathcal{W}}{\partial C_{AB}} F^a{}_A F^b{}_B\,.
\end{aligned} \tag{3}$$

Balance laws. One way to obtain the balance laws for elasticity is through a Lagrangian formulation by using the Hamilton's least action principle. We define the Lagrangian to be a map such that for a motion φ_t of $(\mathcal{B}, \boldsymbol{G}_t)$[2]

$$L(\varphi_t, \dot{\varphi}_t, \boldsymbol{G}_t) = \int_{\mathcal{B}} \mathcal{L}(X, \varphi_t(X), \dot{\varphi}_t(X), \boldsymbol{C}^\flat(X,t), \boldsymbol{G}(X,t))dV(X, \boldsymbol{G})\,,$$

where we assume the Lagrangian density $\mathcal{L} = \mathcal{L}(X, \varphi, \dot{\varphi}, \boldsymbol{C}^\flat, \boldsymbol{G})$ is given by

$$\mathcal{L} = \frac{1}{2}\rho g_{ab}\dot{\varphi}^a\dot{\varphi}^b - \mathcal{W}(X, \boldsymbol{C}^\flat, \boldsymbol{G}) - \mathcal{V}(X, \varphi)\,, \tag{4}$$

where $\mathcal{W} = \mathcal{W}(X, \boldsymbol{C}^\flat, \boldsymbol{G})$ is the elastic energy density defined above and $\mathcal{V} = \mathcal{V}(X, \varphi)$ is the potential energy density. Note that the conservative body force deriving from the potential $\mathcal{V}$ is given by $\boldsymbol{\mathcal{B}}_c = -\frac{1}{\rho}\frac{\partial \mathcal{V}}{\partial \varphi}$. The action functional is defined as

$$S(\varphi, \boldsymbol{G}) = \int_{t_1}^{t_2} L(\varphi_t, \dot{\varphi}_t, \boldsymbol{G}_t)dt\,.$$

In order to take variations, we let φ_ϵ be a 1-parameter family of motions such that[3]

$$\varphi_{0,t} = \varphi_t\,,$$

$$\varphi_{\epsilon,t}\big|_{\partial\mathcal{B}} = \varphi_t\big|_{\partial\mathcal{B}}\,,$$

$$\varphi_{\epsilon,t_1} = \varphi_{t_1}\,, \quad \varphi_{\epsilon,t_2} = \varphi_{t_2}\,.$$

[2]Note that because of the evolution of the reference configuration to account for the evolving source of anelasticity, the material metric $\boldsymbol{G}_t(X) = \boldsymbol{G}(X,t)$ is an independent dynamic variable that should be included in the Lagrangian L.

[3]For fixed t and ϵ, we let $\varphi_{\epsilon,t}(X) := \varphi_\epsilon(X,t)$.

For fixed X and t, we consider the curve $\varphi_{t,X} : \epsilon \to \varphi_{t,X}(\epsilon) := \varphi_{\epsilon,t}(X)$, and define the variation of motion as the spatial vector given by

$$\delta\varphi(X,t) = d_\epsilon \varphi_{t,X}\,[\partial_\epsilon]\Big|_{\epsilon=0}.$$

We also let $\boldsymbol{G}_\epsilon$ be a 1-parameter family of metrics for $\mathcal{B}$ such that $\boldsymbol{G}_0 = \boldsymbol{G}$. We define the variation of the metric tensor $\boldsymbol{G}$ by the material tensor

$$\delta\boldsymbol{G}(X,t) = \frac{d}{d\epsilon}\boldsymbol{G}_\epsilon(X,t)\Big|_{\epsilon=0}.$$

We write the variation of S as a total derivative along the curve $\varphi_{t,X}$ evaluated at $\epsilon = 0$:

$$\delta S(\varphi, \boldsymbol{G}) = \frac{d}{d\epsilon} S(\varphi_\epsilon, \boldsymbol{G}_\epsilon)\bigg|_{\epsilon=0}.$$

For a conservative system, Hamilton's least action principle states that the physical motion φ and the evolution of any other dynamical variable—the material metric $\boldsymbol{G}$ in this case—between t_1 and t_2 is the critical point for the action functional, i.e., the variation of S vanishes at $(\varphi, \boldsymbol{G})$. However, in order to account for possibly non-conservative body forces $\boldsymbol{\mathfrak{B}}_n$, and dissipative forces $\boldsymbol{F}_{\boldsymbol{G}}$ associated with the variation of the material metric $\boldsymbol{G}$, one needs to use a Lagrange-d'Alembert's principle which reads

$$\delta S(\varphi, T) + \int_{t_0}^{t_1}\int_{\mathcal{B}} \left(\rho \boldsymbol{\mathfrak{B}}_n.\delta\varphi + \rho \boldsymbol{F}_{\boldsymbol{G}} \!:\! \delta\boldsymbol{G}\right) dV dt = 0\,.$$

It follows that

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}.\delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{C}^\flat} \!:\! \delta \boldsymbol{C}^\flat + \frac{\partial \mathcal{L}}{\partial \boldsymbol{G}} \!:\! \delta\boldsymbol{G} + \rho\boldsymbol{\mathfrak{B}}_n.\delta\varphi + \rho\boldsymbol{F}_{\boldsymbol{G}} \!:\! \delta\boldsymbol{G}\right) dV dt = 0\,. \tag{5}$$

For different values of ϵ, the velocity vector field $\dot{\varphi}_\epsilon$ lies in different tangent spaces $T_{\varphi_\epsilon(X,t)}\mathcal{S}$. Therefore, the variation of the velocity is given by its covariant derivative along the curve $\varphi_{t,X}$ in $\mathcal{S}$ evaluated at $\epsilon = 0$ [4]

$$\delta\dot{\varphi} = \bar{\nabla}_{\delta\varphi}\dot{\varphi} = \frac{D\dot{\varphi}_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} = \frac{D\delta\varphi}{dt}\,.$$

[4] Note that we use the symmetry lemma, See [17, 56].

Unlike the velocity vector field, the Cauchy-Green deformation tensor field $\boldsymbol{C}^\flat$ lie in the same space when ϵ is varied. Therefore, its variation is given by the total derivative with respect to ϵ evaluated at $\epsilon = 0$

$$\delta \boldsymbol{C}^\flat = \frac{d\boldsymbol{C}^\flat_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\varphi^*_\epsilon \boldsymbol{g}_\epsilon\right)\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\varphi^* \varphi_* \varphi^*_\epsilon \boldsymbol{g}_\epsilon\right)\bigg|_{\epsilon=0} = \varphi^* \boldsymbol{L}_{\delta\varphi} \boldsymbol{g}\,,$$

which in components reads

$$\delta C_{AB} = F^a{}_A g_{ac} \delta\varphi^c{}_{|B} + F^b{}_B g_{bc} \delta\varphi^c{}_{|A}\,.$$

Therefore, it follows from (5) by arbitrariness of $\delta\varphi$ that

$$-\frac{\partial \mathcal{L}}{\partial \varphi^a} - \frac{D}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a}\right) - 2\left(\frac{\partial \mathcal{L}}{\partial C_{AB}} F^b{}_B g_{ab}\right)_{|A} + \rho g_{ab} \mathcal{B}^b_n = 0\,. \tag{6}$$

For the Lagrangian density (4), we have following (2)

$$\frac{1}{\sqrt{\det \boldsymbol{G}}} \frac{D}{dt}\left(\sqrt{\det \boldsymbol{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right)_a = \rho g_{ab} A^b + S_m g_{ab} V^b\,.$$

Therefore, (119) yields the local form of the balance of linear momentum

$$\mathrm{Div}\boldsymbol{P} + \rho \boldsymbol{\mathcal{B}} - S_m \boldsymbol{V} = \rho \boldsymbol{A}\,, \tag{7}$$

where $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}_c + \boldsymbol{\mathcal{B}}_n$ is the total body force per unit undeformed mass. Note, however, that we obtain the local form of the balance of angular momentum as a consequence of the stress constitutive relation and the symmetry of the right Cauchy-Green deformation tensor

$$\boldsymbol{P}\boldsymbol{F}^\mathsf{T} = \boldsymbol{F}\boldsymbol{P}^\mathsf{T}\,.$$

Kinetic equations of evolution On the other hand, by arbitrariness of $\delta\boldsymbol{G}$, it follows from (5) that

$$\frac{\partial \mathcal{W}}{\partial \boldsymbol{G}} = \rho \boldsymbol{F}_{\boldsymbol{G}}\,. \tag{8}$$

If we further assume the existence of a Rayleigh dissipation potential $\mathcal{R} = \mathcal{R}(\boldsymbol{G}, \dot{\boldsymbol{G}})$ such that $\rho \boldsymbol{F}_{\boldsymbol{G}} = -\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}}$, (8) takes the form of a kinetic equation for the evolution of the material metric and reads

$$\frac{\partial \mathcal{W}}{\partial \boldsymbol{G}} = -\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}}\,. \tag{9}$$

2.4 Example: Thermoelasticity

As an application for the proposed geometric theory of anelasticity, we consider the case of thermoelasticity. See our published work on thermoelasticity [70] for more details. First, we detail the construction of the material metric in the case of thermoelasticity. Next, we study the nonlinear thermoelastic problem in a spherical ball. We formulate the governing equations and analytically solve for the thermal stress field for an arbitrary incompressible isotropic hyperelastic solid with a radially-symmetric temperature distribution. Then, we restrict the problem to the thermally isotropic and homogeneous solids following the thermoelastic constitutive model for rubber-like materials described in § 2.4.3 to numerically solve for the static and time-dependent temperature and the thermal stress fields induced by a thermal inclusion. We will also compare the nonlinear solutions with their corresponding linear elasticity solutions.

2.4.1 The material metric and the thermal expansion properties of the material

In the case when the body is subject to a non-uniform temperature field or if the body has inhomogeneous thermal expansion properties and is subject to a uniform temperature field, such a field presents a source of anelasticity and one might construct the material metric to be explicitly dependent on the temperature field $T = T(X,t)$. The distance between two fixed points in $\mathcal{B}$ measured by the material metric $\boldsymbol{G} = \boldsymbol{G}(X,T)$ is equal to the length of the curve resulting from the thermal expansion under the temperature field $T = T(X,t)$ of the line initially connecting them. Therefore, the material manifold $(\mathcal{B}, \boldsymbol{G}(X,T))$ is indeed stress-free by construction. Let us construct such a metric. In order to represent the thermal expansion properties, we introduce three real-valued functions of temperature and position $\{\omega_A(X,T)\}$, $A = 1,2,3$, to describe the thermal expansion properties of the material in the directions $\{\frac{\partial}{\partial Y^A}\}$ at every material point X. We define the temperature-dependent material metric as

$$\boldsymbol{G}(X,T) = \sum_K e^{2\omega_K(X,T)} dY^K \otimes dY^K \,. \tag{10}$$

Let $\zeta_{(K)} : I \to \mathcal{B}$ (where $I \subset \mathbb{R}$ is an interval and $K = 1,2,3$) be a curve in $(\mathcal{B}, \boldsymbol{G})$ such that, in the coordinate chart $\{Y^A\}$, we have $(\zeta_{(K)})^A(s) = (\delta^A{}_K)s$, for $s \in I$. At a point $X = \zeta_{(K)}(s)$, the arc length of the curve $\zeta_{(K)}$ measures the length in the direction $\frac{\partial}{\partial Y^K}$. It is given by (no summation on K)

$$\begin{aligned} dL[\zeta_{(K)}](s,T) =& \sqrt{\boldsymbol{G}(\zeta_{(K)}(s),T)\left(\dot{\zeta}_{(K)}(s), \dot{\zeta}_{(K)}(s)\right)} ds \\ =& \sqrt{G_{KK}(\zeta_{(K)}(s),T)} ds \\ =& e^{\omega_K(X,T)} ds \,. \end{aligned}$$

Therefore

$$\frac{\partial (dL[\zeta_{(K)}])}{\partial T} = \frac{\partial \omega_K}{\partial T} dL[\zeta_{(K)}] \,,$$

and one can read the linear thermal expansion coefficient of the material in the direction $\frac{\partial}{\partial Y^K}$ as

$$\alpha_K(X,T) = \frac{\partial \omega_K}{\partial T}(X,T) \,. \tag{11}$$

Note that for the stress-free temperature field T_0, there is no stretching of the material and hence $\omega_K(X,T_0) = 0$ for $K = 1,2,3$. Therefore, $\boldsymbol{G}(X,T_0) = \boldsymbol{G}_0(X)$. Following (10), one can equivalently represent $\boldsymbol{G}$ in $\{Y^A\}$ as

$$\boldsymbol{G}(X,T) = \sum_{K,L} e^{2\omega_K} \delta^K{}_L \frac{\partial}{\partial Y^K} \otimes dY^L = \sum_K e^{2\omega_K} \frac{\partial}{\partial Y^K} \otimes dY^K \,.$$

Let the change of basis between $\{X^A\}$ and $\{Y^A\}$ be written as

$$dY^K = A^K{}_J dX^J \quad \text{and} \quad \frac{\partial}{\partial Y^K} = (A^{-1})^I{}_K \frac{\partial}{\partial X^I} \,.$$

Then

$$\boldsymbol{G}(X,T) = \left(\sum_K (A^{-1})^I{}_K e^{2\omega_K} A^K{}_J \right) \frac{\partial}{\partial X^I} \otimes dX^J \,.$$

Let $\boldsymbol{\omega}$ be the $\binom{1}{1}$-tensor with the following matrix representation in $\{Y^A\}$

$$\hat{\boldsymbol{\omega}} = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} . \tag{12}$$

Now its representation in $\{X^A\}$ reads off as

$$\omega^I{}_J = \sum_K (A^{-1})^I{}_K \omega_K A^K{}_J \,, \tag{13}$$

and $e^{\boldsymbol{\omega}}$ has the following representation in $\{X^A\}$

$$(e^{2\boldsymbol{\omega}})^I{}_J = \sum_K (A^{-1})^I{}_K e^{2\omega_K} A^K{}_J \,.$$

Therefore

$$\boldsymbol{G}(X,T) = (e^{2\boldsymbol{\omega}})^I{}_J \frac{\partial}{\partial X^I} \otimes dX^J \,.$$

Hence

$$\boldsymbol{G}(X,T) = (G_0)_{IK} (e^{2\boldsymbol{\omega}})^K{}_J dX^I \otimes dX^J \,,$$

where $(G_0)_{IK}$ are the components of $\boldsymbol{G}_0$ in $\{X^A\}$. It finally follows that

$$\boldsymbol{G}(X,T) = \boldsymbol{G}_0(X) e^{2\boldsymbol{\omega}(X,T)} \,. \tag{14}$$

As noted earlier, for the stress-free temperature field $T_0(X)$ we have $\boldsymbol{G}(X,T_0) = \boldsymbol{G}_0(X)$, which corresponds to $\boldsymbol{\omega}(X,T_0(X)) = \boldsymbol{0}$. The Riemannian volume form associated with this metric is

$$dV(X,\boldsymbol{G}) = \sqrt{\det \boldsymbol{G}}\, dX^1 \wedge dX^2 \wedge dX^3 = \sqrt{\det \boldsymbol{G}_0}\, e^{\operatorname{tr}(\boldsymbol{\omega}(X,T))} dX^1 \wedge dX^2 \wedge dX^3 = e^{\operatorname{tr}(\boldsymbol{\omega}(X,T))} dV_0(X) \,, \tag{15}$$

where dV_0 is the Riemannian volume form associated with the metric $\boldsymbol{G}_0$. Thus

$$\frac{d}{dT} dV(X,\boldsymbol{G}) = \frac{\partial(\operatorname{tr}(\boldsymbol{\omega}(X,T)))}{\partial T} e^{\operatorname{tr}(\boldsymbol{\omega}(X,T))} dV_0(X) = \frac{\partial(\operatorname{tr}(\boldsymbol{\omega}(X,T)))}{\partial T} dV(X,T) \,.$$

Therefore, the volumetric thermal expansion coefficient $\beta(X,T)$ of the material reads

$$\beta(X,T) = \frac{\partial}{\partial T} \left[\operatorname{tr}(\boldsymbol{\omega}(X,T)) \right] . \tag{16}$$

If one further assumes that the material is thermally isotropic, the matrix $\boldsymbol{\omega}$ reduces to a scalar function ω times the identity matrix, and the metric reduces to $\boldsymbol{G}(X,T) = \boldsymbol{G}_0(X)e^{2\omega(X,T)}$. The Riemannian volume form associated with this metric is

$$dV(X,\boldsymbol{G}) = e^{3\omega(X,T)}dV_0(X)\,,$$

and the thermal expansion coefficient $\alpha(X,T)$ reads

$$\alpha(X,T) = \frac{1}{3}\beta(X,T) = \frac{\partial\omega(X,T)}{\partial T}\,. \tag{17}$$

Therefore

$$\omega(X,T) = \int_{T_0}^{T} \alpha(X,\tau)d\tau\,. \tag{18}$$

Remark 2.4.1. The material metric $\boldsymbol{G}$ is defined in such a way to include the thermal expansion properties of the material in order to capture any change of shape due to the temperature field. In other words, the geometry of the material manifold explicitly depends on the material thermal expansion properties and the temperature field; it is not purely kinematic. This is in contrast with the material manifold of solids with distributed defects, which is purely kinematic and only depends on the density of defects [90, 91, 92, 94, 93].

2.4.2 The generalized coupled heat equation

In thermoelasticity, the material metric $\boldsymbol{G}$ depends explicitly on the the temperature field. Therefore, the evolution of the material geometry should be governed by the evolution of the temperature field. In what follows, we derive the generalized heat equation starting from the basic principles of thermodynamics within our geometric thermoelasticity framework. Note that this does not contradict the kinetic equation (9), indeed, the Rayleigh potential $\mathcal{R}$ should be such that the kinetic equation (9) leads to the generalized heat equation (30) we set to derive in this section.

The first law of thermodynamics The first law of thermodynamics postulates the existence of a state function, namely the internal energy that satisfies, in the case of a static material metric and in the absence of any mass input/output, the following balance of energy [85, 32, 31]

$$\frac{d}{dt}\int_{\mathcal{U}}\rho\left(\mathcal{E}+\frac{1}{2}\boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})\right)dV=\int_{\mathcal{U}}\rho\left(\boldsymbol{g}(\boldsymbol{B},\boldsymbol{V})+R\right)dV+\int_{\partial\mathcal{U}}\left(\boldsymbol{g}(\boldsymbol{T},\boldsymbol{V})+H\right)dA\,,$$

where $\mathcal{E}$ is the material specific internal energy, $R(X,t)$ is the heat supply per unit undeformed mass and $H(X,\boldsymbol{D}T,\boldsymbol{N},t)$ is the heat flux across a surface with unit normal $\boldsymbol{N}$, where $T=T(X,t)$ denotes the temperature field and $\boldsymbol{D}T=\frac{\partial T}{\partial X^A}dX^A$. However, in the present work, the material metric is a dynamical variable and there is a external mass input/output. Hence, the energy balance should be modified to take into account the rate of change of the energy due to the rate of change of the material metric as well as the mass input/output. The modified energy balance reads

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}}\rho\left(\mathcal{E}+\frac{1}{2}\boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})\right)dV=\int_{\mathcal{U}}\Bigg[\rho\left(\boldsymbol{g}(\boldsymbol{B},\boldsymbol{V})+R+\frac{\partial\mathcal{E}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}\right)&\\+S_m\left(\mathcal{E}+\frac{1}{2}\boldsymbol{g}(\boldsymbol{V},\boldsymbol{V})\right)\Bigg]dV+\int_{\partial\mathcal{U}}\left(\boldsymbol{g}(\boldsymbol{T},\boldsymbol{V})+H\right)dA\,.&\end{aligned}$$

In localized form, the material balance of energy reads

$$\rho\dot{\mathcal{E}}=\boldsymbol{S}\!:\!\boldsymbol{D}-\operatorname{Div}\boldsymbol{Q}+\rho R+\rho\frac{\partial\mathcal{E}}{\partial\boldsymbol{G}}\!:\!\dot{\boldsymbol{G}}\,,\tag{19}$$

where a doted quantity denotes its total time derivative, e.g., $\dot{\mathcal{E}}=\frac{d\mathcal{E}}{dt}$, $\boldsymbol{D}=\frac{1}{2}\dot{\boldsymbol{C}}^\flat(X,t)$ is the material rate of deformation tensor, $\boldsymbol{Q}=\boldsymbol{Q}(X,t)$ is the external heat flux vector per unit area such that $H(X,\boldsymbol{D}T,\boldsymbol{N},t)=-\boldsymbol{g}\left(\boldsymbol{Q}(X,t),\boldsymbol{N}\right)$. In local coordinates, we write the divergence of $\boldsymbol{Q}$ as

$$\operatorname{Div}\boldsymbol{Q}=Q^A{}_{|A}=\frac{\partial Q^A}{\partial X^A}+\Gamma^A_{AB}Q^B\,.$$

The second law of thermodynamics The second law of thermodynamics postulates the existence of a state function, namely the entropy that satisfies, in the case of a static material metric, the material Clausius-Duhem inequality [85, 32, 31]

$$\frac{d}{dt}\int_{\mathcal{U}} \rho \mathcal{N} dV \geq \int_{\mathcal{U}} \rho \frac{R}{T} dV + \int_{\partial\mathcal{U}} \frac{H}{T} dA\,,$$

where $\mathcal{N} = \mathcal{N}(X, T, \boldsymbol{C}^\flat, \boldsymbol{G})$ is the specific entropy. Similarly to the first law of thermodynamics, the Clausius-Duhem inequality should also be modified to include the rate of change of the material metric and the mass input/output

$$\frac{d}{dt}\int_{\mathcal{U}} \rho \mathcal{N} dV \geq \int_{\mathcal{U}} \left[\rho\left(\frac{R}{T} + \frac{\partial \mathcal{N}}{\partial \boldsymbol{G}} \!:\! \dot{\boldsymbol{G}}\right) + S_m \mathcal{N}\right] dV + \int_{\partial\mathcal{U}} \frac{H}{T} dA\,.$$

In localized form, the material Clausius-Duhem inequality reads

$$\rho \dot{\mathcal{N}} \geq \rho\left(\frac{R}{T} + \frac{\partial \mathcal{N}}{\partial \boldsymbol{G}} \!:\! \dot{\boldsymbol{G}}\right) - \operatorname{Div}\left(\frac{\boldsymbol{Q}}{T}\right). \tag{20}$$

Constitutive equations In the following we prove that the constitutive equations (3) can actually be obtained as a consequence of the restrictions imposed by the Clausius-Duhem inequality (20) on the constitutive model. The (hyperelastic) constitutive model is given by the specific free energy function

$$\Psi = \Psi(X, T, \boldsymbol{C}^\flat, \boldsymbol{G})\,,$$

such that the specific internal energy $\mathcal{E}$ is the Legendre transform of $-\Psi$ with respect to the conjugate variables T and $\mathcal{N}$, i.e.

$$\mathcal{E} = T\mathcal{N} + \Psi\,, \tag{21a}$$

$$\mathcal{N} = -\frac{\partial \Psi}{\partial T}\,. \tag{21b}$$

It follows that the specific internal energy $\mathcal{E}$ is such that

$$\mathcal{E} = \mathcal{E}(X, \mathcal{N}, \boldsymbol{C}^\flat, \boldsymbol{G})\,, \quad \frac{\partial \mathcal{E}}{\partial \mathcal{N}} = T\,, \quad \frac{\partial \mathcal{E}}{\partial \boldsymbol{G}} = \frac{\partial \Psi}{\partial \boldsymbol{G}}\,, \quad \frac{\partial \mathcal{E}}{\partial \boldsymbol{C}^\flat} = \frac{\partial \Psi}{\partial \boldsymbol{C}^\flat}\,. \tag{22}$$

By using (19) and (22) in (20), we find

$$\rho \frac{\partial \Psi}{\partial \boldsymbol{C}^\flat} \!:\! \dot{\boldsymbol{C}}^\flat + \rho T \dot{N} - \rho T \frac{\partial \mathcal{N}}{\partial \boldsymbol{C}^\flat} \!:\! \dot{\boldsymbol{C}}^\flat - \rho T \frac{\partial \mathcal{N}}{\partial T} \dot{T} - \boldsymbol{S} \!:\! \boldsymbol{D} + \frac{1}{T} \langle \boldsymbol{D}T, \boldsymbol{Q} \rangle \leq 0\,. \tag{23}$$

Therefore, recalling that $\boldsymbol{D} = \frac{1}{2}\dot{\boldsymbol{C}}^\flat$, we can rewrite (23) as

$$\left(2\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^\flat} - \boldsymbol{S}\right) : \boldsymbol{D} + \rho T\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}} : \dot{\boldsymbol{G}} + \frac{1}{T}\left\langle \boldsymbol{D}T, \boldsymbol{Q}\right\rangle \leq 0\,. \tag{24}$$

The above inequality holds for all deformations φ and metrics $\boldsymbol{G}$. In particular, if we choose φ to be time-independent, (24) reads

$$\rho T\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}} : \dot{\boldsymbol{G}} + \frac{1}{T}\left\langle \boldsymbol{D}T, \boldsymbol{Q}\right\rangle \leq 0\,.$$

Note that $\boldsymbol{D}T$ and $\dot{\boldsymbol{G}} = \frac{d\boldsymbol{G}}{dT}\dot{T}$ can be chosen arbitrarily and independently. Let T be homogeneous, i.e., $\boldsymbol{D}T = \boldsymbol{0}$, then $\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}} : \dot{\boldsymbol{G}} \leq 0$ must hold for every $\dot{\boldsymbol{G}}$. Therefore, we must have

$$\frac{\partial\mathcal{N}}{\partial\boldsymbol{G}} = \boldsymbol{0}\,.$$

Now we assume that temperature is homogeneous and time-independent. Thus, the inequality (24) reads

$$\left(2\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^\flat} - \boldsymbol{S}\right) : \boldsymbol{D} \leq 0\,,$$

and must hold for every $\boldsymbol{D}$. It finally follows that

$$\boldsymbol{S} = 2\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^\flat}\,. \tag{25}$$

Nonlinearly coupled heat equation Following the laws of thermodynamics, one can find the nonlinear coupled heat equation. Note that the energy balance (19) can be simplified to

$$\rho T\dot{\mathcal{N}} = \rho R - \mathrm{Div}\boldsymbol{Q}\,. \tag{26}$$

We substitute (21b) and (25) into (26) and obtain

$$\mathrm{Div}\,\boldsymbol{Q} = \rho\frac{\partial^2\Psi}{\partial T^2}T\dot{T} + \frac{1}{2}T\frac{\partial\boldsymbol{S}}{\partial T} : \dot{\boldsymbol{C}}^\flat + \rho R\,. \tag{27}$$

The specific heat capacity at constant strain c_E is defined as the quantity of heat required to produce a unit temperature increase in a unit mass of material at constant strain ($\dot{\boldsymbol{C}}^\flat = \boldsymbol{0}$), i.e.

$$\mathrm{Div}\,\boldsymbol{Q} = -\rho c_E\dot{T}\,. \tag{28}$$

Comparing (27) and (28) at constant strain and without external heat supply ($R = 0$), we find[5]

$$c_E = -T \frac{\partial^2 \Psi}{\partial T^2} \,. \tag{29}$$

We can now rewrite (27) as the nonlinear coupled heat equation

$$\operatorname{Div} \boldsymbol{Q} = -\rho c_E \dot{T} + \frac{1}{2} T \frac{\partial \boldsymbol{S}}{\partial T} : \dot{\boldsymbol{C}}^\flat + \rho R \,. \tag{30}$$

In particular one can further assume a Fourier's law of thermal conduction, i.e., $\boldsymbol{Q} = -\boldsymbol{K} \cdot \mathrm{D}T$, where $\boldsymbol{K}$ denotes the heat conduction tesor as a positive semi-definite symmetric material $\binom{0}{2}$-tensor.

2.4.3 A nonlinear thermoelastic constitutive model

In the following, we present, in the context of the proposed geometric theory with a temperature-dependent material metric, a thermoelastic model for rubber-like materials following the models proposed by [11], [57, 58, 59], and [35].

For a hyperelastic solid, the free energy provides the material constitutive information given by the independent variables $(X, T, \boldsymbol{C}, \boldsymbol{G})$. The free energy density is defined as[6] $\psi = E - TN$ (E is the internal energy density and N is the entropy density) and the (hyperelastic) constitutive model reads $\psi = \psi(X, T, \boldsymbol{C}, \boldsymbol{G})$. The specific heat capacity at constant strain c_E is defined as (cf. (29))

$$c_E = -T \frac{\partial^2 (\psi/\rho)}{\partial T^2} = T \frac{\partial (N/\rho)}{\partial T} = \frac{\partial (E/\rho)}{\partial T} \,. \tag{31}$$

If we assume that c_E depends only on temperature, then we can write[7]

$$\begin{aligned} E(X, T, \boldsymbol{C}(X,T), \boldsymbol{G}(X,T)) - E(X, T_0, \boldsymbol{C}(X,T), \boldsymbol{G}(X,T)) = \rho(X, \boldsymbol{G}(X,T)) \\ \times \int_{T_0}^{T} c_E(\tau) d\tau \,, \end{aligned} \tag{32}$$

[5]Note that the partial derivative with respect to temperature in (27) and (29) is a partial derivative with respect to temperature with $\boldsymbol{C}$ and $\boldsymbol{G}$ being fixed, i.e., $\frac{\partial}{\partial T} = \frac{\partial}{\partial T}|_{\boldsymbol{C},\boldsymbol{G}}$.

[6]Note that $\psi = \rho \Psi$.

[7]Note that the material mass density ρ depends only implicitly on temperature via the material metric, i.e., $\frac{\partial \rho}{\partial T} = 0$ but $\frac{d\rho}{dT} = -\beta \rho$.

and

$$N(X,T,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) - N(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) = \rho(X,\boldsymbol{G}(X,T)) \times \int_{T_0}^{T} c_E(\tau)\frac{d\tau}{\tau}\,. \tag{33}$$

We can therefore write

$$\begin{aligned}\psi(X,T,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) &= -\left(\frac{T}{T_0}-1\right) E(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) \\ &+\frac{T}{T_0}\psi(X,T_0,\boldsymbol{C}(X,T),\boldsymbol{G}(X,T)) - \rho(X,\boldsymbol{G}(X,T))\int_{T_0}^{T} c_E(\tau)\frac{T-\tau}{\tau}d\tau\,.\end{aligned} \tag{34}$$

In the case of an isotropic solid, the specific free energy ψ depends on $\mathrm{I}=\mathrm{tr}(\boldsymbol{C})$, $\mathrm{II}=\det(\boldsymbol{C})\,\mathrm{tr}(\boldsymbol{C}^{-1})=\frac{1}{2}(\mathrm{tr}(\boldsymbol{C})^2-\mathrm{tr}(\boldsymbol{C}^2))$, and $J=\sqrt{\frac{\det \boldsymbol{C}^\flat}{\det \boldsymbol{G}}}$, which are the principal invariants of $\boldsymbol{C}$. Furthermore, experiments suggest that for rubber-like materials, the internal energy density depends only on the volumetric part of deformation [83], i.e, we can write $E=E(T,J)$. Note that this confirms the assumption made on $c_E=c_E(T)$. Following the works cited above, if we denote by κ_0, μ_0 and β_0, the bulk modulus, the shear modulus, and the volumetric coefficient of thermal expansion at T_0, respectively, we consider the following constitutive model for a homogeneous isotropic rubber-like material

$$\psi(T_0,\tilde{\mathrm{I}}(X,T),J(X,T)) = \frac{\mu_0}{2}(\tilde{\mathrm{I}}-3)+\frac{\kappa_0}{2}(J-1)^2, \quad E(T_0,J)=\kappa_0\beta_0T_0(J-1), \tag{35}$$

where $\tilde{\mathrm{I}}=J^{-2/3}\mathrm{I}$. It follows that

$$\psi(T,\tilde{\mathrm{I}},J) = \frac{\mu_0}{2}\frac{T}{T_0}(\tilde{\mathrm{I}}-3)+\frac{\kappa_0}{2}\frac{T}{T_0}(J-1)^2-\kappa_0\beta_0(J-1)\,(T-T_0)-\rho\int_{T_0}^{T} c_E(\tau)\frac{T-\tau}{\tau}d\tau\,. \tag{36}$$

In the incompressible case, we have the constraint $J-1=0$ associated with the pressure field p as the Lagrange multiplier

$$\psi(T,\mathrm{I},J) = \frac{\mu_o}{2}\frac{T}{T_o}(\mathrm{I}-3)-\rho\int_{T_o}^{T} c_E(\tau)\frac{T-\tau}{\tau}d\tau - p(J-1)\,. \tag{37}$$

One may now ask if it is possible to find a relation between the function $\boldsymbol{\omega}(T)$ appearing in the material metric and the free energy (36). The answer is affirmative.

Let us consider a homogeneous body modeled by the free energy density (36) and assume that it is stress free at the uniform temperature T_0. Now let us assume that the temperature of the body is changed to another uniform temperature T. The body undergos a purely volumetric deformation and remains stress free. Note that the mean Cauchy stress $\sigma = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$ is given by

$$\sigma = \frac{\partial\psi}{\partial J} = \kappa_0 \frac{T}{T_0}(J-1) - \kappa_0\beta_0(T-T_0) = 0. \tag{38}$$

Therefore

$$J = 1 + \beta_0 T_0 \left(1 - \frac{T_0}{T}\right). \tag{39}$$

On the other hand, note that we have for this deformation

$$J = e^{\mathrm{tr}(\boldsymbol{\omega}(T))}. \tag{40}$$

It follows from (39) and (40) that

$$\mathrm{tr}(\boldsymbol{\omega}(T)) = \ln\left[1 + \beta_0 T_0 \left(1 - \frac{T_0}{T}\right)\right], \tag{41}$$

and hence[8]

$$\beta(T) = \frac{\beta_0 \frac{T_0^2}{T^2}}{1 + \beta_0 T_0 \left(1 - \frac{T_0}{T}\right)}. \tag{42}$$

2.4.4 A spherical ball made of an incompressible isotropic solid

In this section we consider an incompressible isotropic solid sphere of radius R_o under uniform normal traction on its boundary and ignore body forces. In spherical coordinates (R,Θ,Φ), for which $R \geq 0$, $0 \leq \Theta \leq \pi$, and $0 \leq \Phi < 2\pi$, the material metric for the configuration with the stress-free temperature field $T_0 = T_0(R)$, reads

$$\boldsymbol{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\Theta \end{pmatrix}. \tag{43}$$

[8]Note that we recover the result derived by [59], Eq. (102-c).

We assume a radially-symmetric temperature field $T = T(R,t)$ in the ball and let $\alpha_R = \alpha_R(R,T)$ be the radial thermal expansion coefficient and $\alpha_\Theta = \alpha_\Theta(R,T)$ be the circumferential thermal expansion coefficient of the ball. The temperature-dependent material metric of the ball reads

$$\boldsymbol{G} = \begin{pmatrix} e^{2\omega_R} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta} & 0 \\ 0 & 0 & R^2 e^{2\omega_\Theta}\sin^2\Theta \end{pmatrix}, \tag{44}$$

where for $K \in \{R,\Theta\}$, $\omega_K(R,T(R,t)) = \displaystyle\int_{T_0}^{T(R,t)} \alpha_K(R,\tau)d\tau$. The Christoffel symbol matrices of $\boldsymbol{G}$ read

$$\left[\Gamma^R{}_{AB}\right] = \begin{pmatrix} \frac{\partial\omega_R}{\partial R} & 0 & 0 \\ 0 & -e^{2(\omega_\Theta-\omega_R)}(R+R^2\frac{\partial\omega_\Theta}{\partial R}) & 0 \\ 0 & 0 & -e^{2(\omega_\Theta-\omega_R)}\left(R+R^2\frac{\partial\omega_\Theta}{\partial R}\right)\sin^2\Theta \end{pmatrix},$$

$$\left[\Gamma^\Theta{}_{AB}\right] = \begin{pmatrix} 0 & \frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R} & 0 \\ \frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R} & 0 & 0 \\ 0 & 0 & -\sin\Theta\cos\Theta \end{pmatrix},$$

$$\left[\Gamma^\Phi{}_{AB}\right] = \begin{pmatrix} 0 & 0 & \frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R} \\ 0 & 0 & 1/\tan\Theta \\ \frac{1}{R}+\frac{\partial\omega_\Theta}{\partial R} & 1/\tan\Theta & 0 \end{pmatrix}, \tag{45}$$

where $\frac{\partial\omega_K}{\partial R} = \frac{\partial\omega_K}{\partial R}\big|_t = \frac{\partial\omega_K}{\partial R}\big|_{t,T} + \alpha_K\frac{\partial T}{\partial R}$. We equip the ambient space with the following flat metric in the spherical coordinates (r,θ,ϕ).

$$\boldsymbol{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{46}$$

The Christoffel symbol matrices for $\boldsymbol{g}$ read

$$[\gamma^r{}_{ab}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{pmatrix}, \quad \left[\gamma^\theta{}_{ab}\right] = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{pmatrix},$$
$$\left[\gamma^\phi{}_{ab}\right] = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & 1/\tan\Theta \\ \frac{1}{r} & 1/\tan\Theta & 0 \end{pmatrix}. \tag{47}$$

Thermal stresses. We next solve for the thermal stress field when the solid sphere is made of an arbitrary incompressible isotropic solid and the temperature field is radially symmetric. In order to calculate the thermal stresses, we embed the material manifold into the ambient space and look for solutions of the form $(r,\theta,\phi) = (r(R,t),\Theta,\Phi)$. The deformation gradient reads

$$\boldsymbol{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{48}$$

For an incompressible solid, we have

$$J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}}\det \boldsymbol{F} = \frac{r^2}{R^2 e^{\omega_R + 2\omega_\Theta}} r' = 1\,, \tag{49}$$

and hence, assuming $r(0,T) = 0$ (to eliminate rigid translations), we find

$$r(R,t) = \left(\int_0^R 3\xi^2 e^{\omega_R(\xi,T(\xi,t)) + 2\omega_\Theta(\xi,T(\xi,t))} d\xi\right)^{\frac{1}{3}}. \tag{50}$$

The right Cauchy-Green deformation tensor reads

$$\boldsymbol{C} = \begin{pmatrix} \frac{R^4}{r^4}e^{4\omega_\Theta} & 0 & 0 \\ 0 & \frac{r^2}{R^2}e^{-2\omega_\Theta} & 0 \\ 0 & 0 & \frac{r^2}{R^2}e^{-2\omega_\Theta} \end{pmatrix}. \tag{51}$$

For an incompressible isotropic solid, the free energy density per unit undeformed volume $\psi = \rho\Psi$ is expressed as a function of $\mathrm{I} = \mathrm{tr}\boldsymbol{C}$ and $\mathrm{II} = \det\boldsymbol{C}\mathrm{tr}\boldsymbol{C}^{-1} = \frac{1}{2}(\mathrm{tr}(\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2))$, i.e., $\psi = \psi(R, T, \mathrm{I}, \mathrm{II}, J = 1)$. Therefore, we can write [18]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B \left[(\psi_{\mathrm{I}} + \mathrm{I}\psi_{\mathrm{II}})\, G^{AB} - \psi_{\mathrm{II}} C^{AB} \right] - p g^{ab} , \tag{52}$$

where $\psi_{\mathrm{I}} = \frac{\partial \psi}{\partial \mathrm{I}}$, $\psi_{\mathrm{II}} = \frac{\partial \psi}{\partial \mathrm{II}}$, and $p = p(R, t)$ is the pressure field due to the incompressibility constraint. Thus, the non-zero components of the Cauchy stress tensor are given by

$$\sigma^{rr} = 2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) - p , \tag{53a}$$

$$\sigma^{\theta\theta} = \frac{1}{r^2}\left\{ 2\frac{r^2 e^{-2\omega_\Theta}}{R^2}\left[\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} + \frac{R^4 e^{4\omega_\Theta}}{r^4}\right)\right] - p \right\} , \tag{53b}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\theta}\sigma^{\theta\theta} , \tag{53c}$$

where $p = p(R, t)$ is the pressure field due to the incompressibility constraint. The only non-trivial equilibrium equation is $\sigma^{ra}{}_{|a} = 0$, which is simplified to read

$$\frac{r^2 e^{-\omega_R - 2\omega_\Theta}}{R^2}\sigma^{rr}{}_{,R} + \frac{2}{r}\sigma^{rr} - 2r\sigma^{\theta\theta} = 0 . \tag{54}$$

This yields

$$\frac{\partial p}{\partial R} = \frac{\partial}{\partial R}\left[2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)\right] - 4\frac{e^{\omega_R}}{r}\left(1 - \frac{R^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) . \tag{55}$$

Assuming that the boundary of the solid sphere is under uniform normal traction, i.e., $\sigma^{rr}(R_o, T(R_o)) = -\sigma_o$, the pressure field at the boundary is

$$p(R_o) = \left[2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)\right]\Bigg|_{R=R_o} + \sigma_o , \tag{56}$$

and it follows that

$$\begin{aligned} p(R, t) = {} & 2\frac{R^4 e^{4\omega_\Theta}}{r^4}\left(\psi_{\mathrm{I}} + 2\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right) \\ & + \int_R^{R_o} 4\frac{e^{\omega_R}}{r}\left(1 - \frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{\xi^2}\right) d\xi + \sigma_o . \end{aligned} \tag{57}$$

Finally, given a radially-symmetric temperature field $T = T(R,t)$, the thermal stress field is given in terms of the non-zero components of the Cauchy stress tensor as

$$\sigma^{rr} = -4\int_R^{R_o} \frac{e^{\omega_R}}{r}\left(1-\frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{\xi^2}\right)d\xi - \sigma_o\,, \tag{58a}$$

$$\sigma^{\theta\theta} = 2\frac{1}{r^2}\left(\frac{r^2 e^{-2\omega_\Theta}}{R^2}-\frac{R^4 e^{4\omega_\Theta}}{r^4}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)+\frac{1}{r^2}\sigma^{rr}\,, \tag{58b}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\theta}\sigma^{\theta\theta}\,. \tag{58c}$$

Alternatively, the non-zero components of the second Piola-Kirchhoff stress tensor read

$$S^{RR} = -\frac{r^4 e^{-4\omega_\Theta-2\omega_R}}{R^4}\left[4\int_R^{R_o}\frac{e^{\omega_R}}{r}\left(1-\frac{\xi^6 e^{6\omega_\Theta}}{r^6}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{\xi^2}\right)d\xi+\sigma_o\right]\,, \tag{59a}$$

$$S^{\Theta\Theta} = \frac{2}{r^2}\left(\frac{r^2}{R^2 e^{6\omega_\Theta}}-\frac{R^4 e^{4\omega_\Theta}}{r^4}\right)\left(\psi_{\mathrm{I}}+\psi_{\mathrm{II}}\frac{r^2 e^{-2\omega_\Theta}}{R^2}\right)+\frac{R^4 e^{4\omega_\Theta+2\omega_R}}{r^6}S^{RR}\,, \tag{59b}$$

$$S^{\Phi\Phi} = \frac{1}{\sin^2\Theta}S^{\Theta\Theta}\,. \tag{59c}$$

Heat equation. We assume that there is no external heat supply, i.e., $R = 0$, and that the heat conduction in the material is isotropic, i.e., $\boldsymbol{K} = k\boldsymbol{G}^\sharp$, where $k = k(R, T(R,t))$ is a scalar valued function. Therefore, the coupled heat equation is rewritten as

$$\begin{aligned}\left[\frac{\partial^2 T}{\partial R^2}+\left(\frac{2}{R}+\frac{1}{k}\left.\frac{\partial k}{\partial R}\right|_t\right)\frac{\partial T}{\partial R}+\left.\frac{\partial\left(2\omega_\Theta-\omega_R\right)}{\partial R}\right|_t\frac{\partial T}{\partial R}\right]ke^{-2\omega_R} &= \rho_o c_E \dot{T} e^{-\omega_R-2\omega_\Theta}\\ &\quad-\frac{1}{2}T\frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,,\end{aligned} \tag{60}$$

where $\rho_o(X) = \rho(X, \boldsymbol{G}_o)$ is the mass density in the stress-free configuration with uniform temperature T_0. If we further assume that the material is thermally isotropic (i.e., $\omega_R = \omega_\Theta = \omega(R, T(R,t))$), (60) reduces to

$$\left[\frac{k}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\left(\left.\frac{\partial k}{\partial R}\right|_t+k\left.\frac{\partial \omega}{\partial R}\right|_t\right)\frac{\partial T}{\partial R}\right]e^{\omega} = \rho_o c_E\dot{T}-\frac{e^{3\omega}}{2}T\frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,. \tag{61}$$

In order to solve for the temperature and stress fields, one needs to find ω_R and ω_Θ to explicitly specify the material metric (cf. (44)). We assume in the remainder of this section that the material is thermally homogeneous and that the metric $\boldsymbol{G}_0$ corresponds to the uniform stress-free temperature field of the ball $T_0(R) = T_o$ (i.e., $\boldsymbol{G}|_{T=T_o} = \boldsymbol{G}_0$), where T_o is the temperature of the outside medium surrounding the ball. For $K \in \{R, \Theta\}$, we have

$$\omega_K(R, T(R,t)) = \int_{T_o}^{T(R,t)} \alpha_K(T) dT \,. \tag{62}$$

Example 2.4.1. In this example we solve for the stress field induced by a static thermal inclusion in a homogeneous, isotropic, thermally homogeneous and anisotropic ball. Next, we assume a thermally isotropic material to compare with the linearized elasticity solution. We consider a thermal inclusion of radius $R_i < R_o$:

$$T(R) = \begin{cases} T_i & R \leq R_i \,, \\ T_o & R \geq R_i \,. \end{cases} \tag{63}$$

Thus, for $K \in \{R, \Theta\}$, we have

$$\omega_K(R) = \begin{cases} \displaystyle\int_{T_o}^{T_i} \alpha_K(T) dT & R \leq R_i \,, \\ 0 & R > R_i \,. \end{cases} \tag{64}$$

We substitute (64) into (50) to find

$$r(R) = \begin{cases} R e^{\frac{1}{3}(\omega_R(R_i) + 2\omega_\Theta(R_i))} & R \leq R_i \,, \\ \left[R_i^3 \left(e^{(\omega_R(R_i) + 2\omega_\Theta(R_i))} - 1 \right) + R^3 \right]^{\frac{1}{3}} & R > R_i \,. \end{cases} \tag{65}$$

Following (58), the physical components of the thermal stress field read

$$\hat{\sigma}^{rr} = -\int_R^{R_o} 4 \frac{e^{\omega_R}}{r} \left(1 - \frac{\xi^6 e^{6\omega_\Theta}}{r^6} \right) \left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} \frac{r^2 e^{-2\omega_\Theta}}{\xi^2} \right) d\xi - \sigma_o \,, \tag{66a}$$

$$\hat{\sigma}^{\theta\theta} = 2 \left(\frac{r^2 e^{-2\omega_\Theta}}{R^2} - \frac{R^4 e^{4\omega_\Theta}}{r^4} \right) \left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} \frac{r^2 e^{-2\omega_\Theta}}{R^2} \right) + \hat{\sigma}^{rr} \,, \tag{66b}$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta} \,. \tag{66c}$$

For $R \leq R_i$, we have

$$\boldsymbol{C} = \begin{pmatrix} e^{-\frac{4}{3}(\omega_R(R_i) - \omega_\Theta(R_i))} & 0 & 0 \\ 0 & e^{\frac{2}{3}(\omega_R(R_i) - \omega_\Theta(R_i))} & 0 \\ 0 & 0 & e^{\frac{2}{3}(\omega_R(R_i) - \omega_\Theta(R_i))} \end{pmatrix} . \tag{67}$$

Therefore, I and II are both constant inside the inclusion, and because the material is homogeneous and isotropic, $\psi = \psi(T_i, \mathrm{I}, \mathrm{II})$. Thus, the terms ψ_{I} and ψ_{II} are both constant and it follows that for $R \leq R_i$, we have

$$\begin{aligned}
\hat{\sigma}^{rr} &= 4e^{\frac{2}{3}(\omega_R(R_i)+2\omega_\Theta(R_i))}\left(e^{-2\omega_\Theta(R_i)} - e^{-2\omega_R(R_i)}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} e^{\frac{2}{3}(\omega_R(R_i)-\omega_\Theta(R_i))}\right) \\
&\quad \times \ln\left(\frac{R}{R_i}\right) + \hat{\sigma}_c\,, \qquad (68a)\\
\hat{\sigma}^{\theta\theta} &= 2e^{\frac{2}{3}(\omega_R(R_i)+2\omega_\Theta(R_i))}\left(e^{-2\omega_\Theta(R_i)} - e^{-2\omega_R(R_i)}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}} e^{\frac{2}{3}(\omega_R(R_i)-\omega_\Theta(R_i))}\right) \\
&\quad \times \left[2\ln\left(\frac{R}{R_i}\right) + 1\right] + \hat{\sigma}_c\,, \qquad (68b)\\
\hat{\sigma}^{\phi\phi} &= \hat{\sigma}^{\theta\theta}\,,
\end{aligned}$$

where $\hat{\sigma}_c$ is a constant given by

$$\hat{\sigma}_c = -4\int_{R_i}^{R_o} \frac{1}{r(\xi)}\left(1 - \frac{\xi^6}{r^6(\xi)}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2(\xi)}{\xi^2}\right) d\xi - \sigma_o\,. \qquad (69)$$

On the other hand, for $R > R_i$, we have

$$\hat{\sigma}^{rr} = -\int_{R}^{R_o} \frac{4}{r(\xi)}\left(1 - \frac{\xi^6}{r^6(\xi)}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2(\xi)}{\xi^2}\right) d\xi - \sigma_o\,, \qquad (70a)$$

$$\hat{\sigma}^{\theta\theta} = 2\left(\frac{r^2(R)}{R^2} - \frac{R^4}{r^4(R)}\right)\left(\psi_{\mathrm{I}} + \psi_{\mathrm{II}}\frac{r^2(R)}{R^2}\right) + \hat{\sigma}^{rr}\,, \qquad (70b)$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}\,. \qquad (70c)$$

Remark 2.4.2. Note that in the absence of body forces, for a homogeneous, isotropic, thermally homogeneous and isotropic solid sphere with a uniform normal traction on its boundary and with a radially-symmetric thermal inclusion, the stress inside the inclusion is uniform and hydrostatic. However, if the material is thermally anisotropic (i.e., $\omega_R \neq \omega_\Theta$), the stress field has a logarithmic singularity at the center of the ball.[9]

Comparison with the linear case. Next, we compare the thermal stress field with the classical linear elasticity solution. We consider a homogeneous, isotropic,

[9] Similar results were observed for distributed eigenstrains in [92, 95].

thermally homogeneous and isotropic, and traction-free solid sphere. The classical linear elasticity solution of the sphere problem for constant μ_o and α_o reads [7, 33]

$$\hat{\sigma}^{rr} = 4\mu_o\alpha_o\frac{3\lambda+2\mu_o}{\lambda+2\mu_o}\left[\frac{1}{R_o^3}\int_0^{R_o}T(\xi)\xi^2 d\xi - \frac{1}{R^3}\int_0^{R}T(\xi)\xi^2 d\xi\right], \tag{71a}$$

$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\mu_o\alpha_o\frac{3\lambda+2\mu_o}{\lambda+2\mu_o}\left[\frac{2}{R_o^3}\int_0^{R_o}T(\xi)\xi^2 d\xi + \frac{1}{R^3}\int_0^{R}T(\xi)\xi^2 d\xi - T\right]. \tag{71b}$$

Incompressible linearized elasticity corresponds to $\nu = 0.5$, i.e., $\lambda \to \infty$. Thus, in the case of the thermal inclusion (63), we find

$$\begin{aligned} R \leq R_i &: \quad \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = -4\mu_o\left(1-\frac{R_i^3}{R_o^3}\right)\alpha_o\Delta_i T, \\ R > R_i &: \quad \begin{cases} \hat{\sigma}^{rr} = -4\mu_o\left(\dfrac{R_i^3}{R^3}-\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_i T, \\ \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\mu_o\left(\dfrac{R_i^3}{R^3}+2\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_i T, \end{cases} \end{aligned} \tag{72}$$

where $\Delta_i T = T_i - T_o$.

In order to compare the nonlinear solution with the linearized one, we consider the thermoelastic model presented in § 2.4.3 and enforce the incompressibility condition. Hence, following (41), we find for the thermal inclusion (63)

$$\omega(R) = \begin{cases} \dfrac{1}{3}\ln\left[\dfrac{(1+3\alpha_o T_o)\Delta_i T + T_o}{\Delta_i T + T_o}\right] & R \leq R_i, \\ 0 & R > R_i, \end{cases} \tag{73}$$

and from (66) and (37) the thermal stress field reads

$$\begin{aligned} R \leq R_i &: \quad \left\{ \hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = -2\mu_o\int_{R_i}^{R_o}\frac{1}{r(\xi)}\left(1-\frac{\xi^6}{r^6(\xi)}\right)d\xi, \right. \\ R > R_i &: \quad \begin{cases} \hat{\sigma}^{rr} = -2\mu_o\displaystyle\int_{R}^{R_o}\frac{1}{r(\xi)}\left(1-\frac{\xi^6}{r^6(\xi)}\right)d\xi, \\ \hat{\sigma}^{\theta\theta} = \mu_o\left(\dfrac{r^2(R)}{R^2}-\dfrac{R^4}{r^4(R)}\right)+\hat{\sigma}^{rr}, \\ \hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}, \end{cases} \end{aligned} \tag{74}$$

where

$$r(R)=\begin{cases}\left[\dfrac{(1+3\alpha_oT_o)\Delta_iT+T_o}{\Delta_iT+T_o}\right]^{\frac{1}{3}}R & R\leq R_i\,,\\ \left[3\alpha_oT_o\dfrac{\Delta_iT}{\Delta_iT+T_o}R_i^3+R^3\right]^{\frac{1}{3}} & R\geq R_i\,.\end{cases}\tag{75}$$

For small Δ_iT , we have the following asymptotic expansions:

$$R\leq R_i:$$

$$\begin{cases}\hat{\sigma}^{rr}=\hat{\sigma}^{\theta\theta}=\hat{\sigma}^{\phi\phi}=-4\mu_o\left(1-\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_iT\\ \quad+\left(1-\dfrac{R_i^3}{R_o^3}\right)\left[4+11\left(1+\dfrac{R_i^3}{R_o^3}\right)\alpha_oT_o\right]\dfrac{\alpha_o}{T_o}\left(\Delta_iT\right)^2+o\left(\left(\Delta_iT\right)^3\right)\,,\end{cases}$$

$$R>R_i:$$

$$\begin{cases}\hat{\sigma}^{rr}=-4\mu_o\left(\dfrac{R_i^3}{R^3}-\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_iT\\ \quad-\mu_o\left(\dfrac{R_i^3}{R^3}-\dfrac{R_i^3}{R_o^3}\right)\left[4+11\left(\dfrac{R_i^3}{R^3}+\dfrac{R_i^3}{R_o^3}+1\right)\alpha_oT_o\right]\dfrac{\alpha_o}{T_o}\left(\Delta_iT\right)^2+o\left(\left(\Delta_iT\right)^3\right)\,,\\ \hat{\sigma}^{\theta\theta}=2\mu_o\left(\dfrac{R_i^3}{R^3}+2\dfrac{R_i^3}{R_o^3}\right)\alpha_o\Delta_iT\\ \quad-\mu_o\left[2\left(\dfrac{R_i^3}{R^3}+\dfrac{R_i^3}{R_o^3}\right)+\left(4\dfrac{R_i^6}{R^6}+11\dfrac{R_i^6}{R_o^6}\right)\alpha_oT_o\right]\dfrac{\alpha_o}{T_o}\left(\Delta_iT\right)^2+o\left(\left(\Delta_iT\right)^3\right)\,,\\ \hat{\sigma}^{\phi\phi}=\hat{\sigma}^{\theta\theta}\,.\end{cases}\tag{76}$$

We have thus recovered, up to the first order in Δ_iT , the classical linearized elasticity solution.

We consider the case of rubber-like solids for which we typically have $\alpha_o=6\times10^{-4}\,\mathrm{K}^{-1}$ at $300°\mathrm{K}$, i.e., $\alpha_oT_o=0.18$. In Figures 1 and 2, we plot the static thermal stresses for different values of the initial relative temperature difference $\delta_T=\frac{\Delta_iT}{T_o}$ in the thermal inclusion (63). The two solutions for the stress field are very close for small values of δ_T (i.e., in the range of validity of linearized elasticity). For larger values of δ_T , even though linearized elasticity captures the overall behavior of $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$, it fails by overestimating their values (the relative difference of stress reaches 45% inside the inclusion for $\delta_T=30\%$).

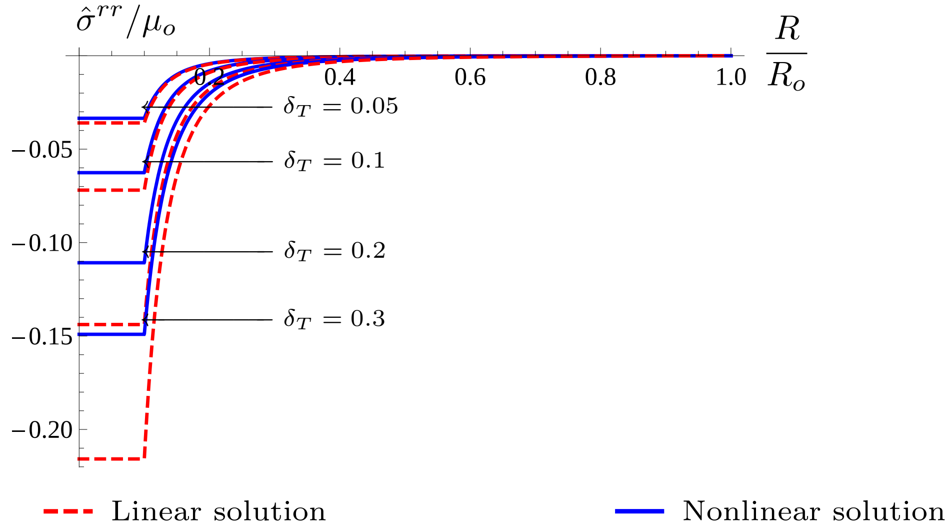


Figure 1: Nonlinear and linear solutions for the radial stress field for $\frac{R_i}{R_o} = 0.1$, $\alpha_o T_o = 0.18$ and different values of $\delta_T = \frac{\Delta_i T}{T_o}$.

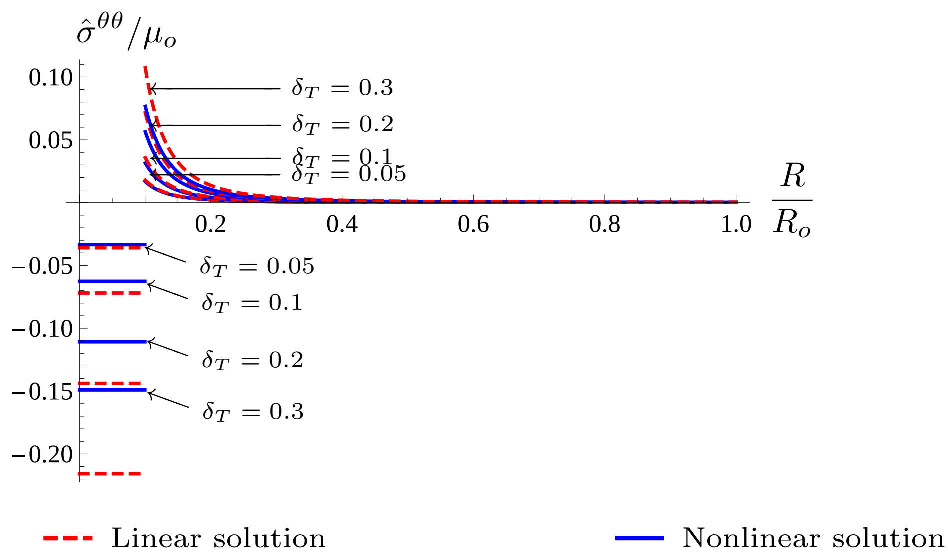


Figure 2: Nonlinear and linear solutions for the circumferential stress field for $\frac{R_i}{R_o} = 0.1$, $\alpha_o T_o = 0.18$ and different values of $\delta_T = \frac{\Delta_i T}{T_o}$.

Example 2.4.2. In this example we numerically solve for the evolution of temperature and thermal stress fields for a homogeneous, isotropic, thermally homogeneous and isotropic, and traction-free ball for which we assume the thermoelastic model described in Appendix 2.4.3. Following (41), we find

$$\omega(R,T(R,t)) = \frac{1}{3}\ln\left[1+3\alpha_0 T_0\left(1-\frac{T_0}{T(R,t)}\right)\right]. \tag{77}$$

Thus

$$\begin{aligned} r(R,t) &= \left\{\int_0^R 3\xi^2\left[1+3\alpha_0 T_0\left(1-\frac{T_0}{T(\xi,t)}\right)\right]d\xi\right\}^{\frac{1}{3}} \\ &= \left[\left(1+3\alpha_0 T_0\right)R^3 - 9\alpha_0 T_0\int_0^R \frac{T_0\xi^2}{T(\xi,t)}d\xi\right]^{\frac{1}{3}}. \end{aligned} \tag{78}$$

Given the free energy density (37), it follows from (58) that the physical components of the Cauchy stress field are

$$\hat{\sigma}^{rr} = -2\mu_o\int_R^{R_o}\frac{\left(1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right)^{1/3}}{r}\left(1-\frac{\xi^6\left[1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right]^2}{r^6}\right)\frac{T}{T_o}d\xi - \sigma_o\,, \tag{79a}$$

$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{rr} + \mu_o\left[\frac{r^2\left(1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right)^{-2/3}}{R^2} - \frac{R^4\left[1+3\alpha_0 T_0\left(1-\frac{T_0}{T}\right)\right]^{4/3}}{r^4}\right]\frac{T}{T_o}\,, \tag{79b}$$

$$\hat{\sigma}^{\phi\phi} = \hat{\sigma}^{\theta\theta}\,. \tag{79c}$$

Now, let us find the time-dependent temperature field in order to evaluate the thermal stress field by solving the coupled heat equation (61). We assume that the heat conduction coefficient depends only on temperature and consider the following empirical model for elastomer vulcanizates (cf. [77]), suggesting that the heat conduction coefficient decreases with temperature

$$k(T(R,t)) = k_o\left[1 - s(T(R,t)-T_o)\right], \tag{80}$$

where $k_o = k(T_o)$ and s is a softening parameter. Therefore, (61) reads

$$\begin{aligned}\left\{[1-s(T-T_o)]\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\alpha\left(\frac{\partial T}{\partial R}\right)^2\right]-s\left(\frac{\partial T}{\partial R}\right)^2\right\}e^{\omega} &= \frac{\rho_o c_E}{k_o}\dot{T} \\ &\quad -\frac{e^{3\omega}}{2k_o}T\frac{\partial S^{AB}}{\partial T}\dot{C}_{AB}\,.\end{aligned} \tag{81}$$

Following (59), the non-zero components of the second Piola-Kirchhoff tensor are

$$S^{RR} = -2\mu_o\frac{r^4e^{-6\omega}}{R^4}\int_R^{R_o}\frac{T}{T_o}\frac{e^{\omega(\xi,T(\xi,t))}}{r(\xi,t)}\left(1-\frac{\xi^6e^{6\omega(\xi,T(\xi,t))}}{r^6(\xi,t)}\right)d\xi\,, \tag{82a}$$

$$S^{\Theta\Theta} = \frac{R^4e^{6\omega}}{r^6}S^{RR}+\mu_o\frac{T}{T_o}\frac{1}{r^2}\left(\frac{r^2}{R^2e^{2\omega}}-\frac{R^4e^{4\omega}}{r^4}\right)\,, \tag{82b}$$

$$S^{\Phi\Phi} = \frac{1}{\sin^2\Theta}S^{\Theta\Theta}\,. \tag{82c}$$

Hence[10]

$$\frac{\partial S^{RR}}{\partial T} = 0\,, \tag{83a}$$

$$\frac{\partial S^{\Theta\Theta}}{\partial T} = \frac{\mu_o}{T_o}\left(\frac{1}{R^2e^{2\omega}}-\frac{R^4e^{4\omega}}{r^6}\right)\,, \tag{83b}$$

$$\frac{\partial S^{\Phi\Phi}}{\partial T} = \frac{1}{\sin^2\Theta}\frac{\partial S^{\Theta\Theta}}{\partial T}\,. \tag{83c}$$

The non-vanishing components of $\dot{\boldsymbol{C}}$ are

$$\dot{C}_{RR} = \frac{\partial}{\partial t}\left(\frac{R^4e^{6\omega}}{r^4}\right) = 2\frac{R^4e^{6\omega}}{r^4}\left(3\alpha\dot{T}-\frac{2}{r}\frac{\partial r}{\partial t}\right)\,, \tag{84a}$$

$$\dot{C}_{\Theta\Theta} = \frac{\partial r^2}{\partial t} = 2r\frac{\partial r}{\partial t}\,, \tag{84b}$$

$$\dot{C}_{\Phi\Phi} = \dot{C}_{\Theta\Theta}\sin^2\Theta\,, \tag{84c}$$

and following (78), we find

$$\frac{\partial r}{\partial t} = \frac{3\alpha_o}{r^2}\int_0^R\xi^2\left(\frac{T_o}{T(\xi,t)}\right)^2\dot{T}(\xi,t)d\xi\,. \tag{85}$$

[10]Recall, as noted earlier, that $\frac{\partial}{\partial T} = \frac{\partial}{\partial T}|_{\boldsymbol{C},\boldsymbol{G}}$ is a partial derivative with respect to T with $\boldsymbol{C}$ and $\boldsymbol{G}$ fixed.

Thus, the coupled heat equation (61) reads

$$\left\{[1-s(T-T_o)]\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial T}{\partial R}\right)+\alpha\left(\frac{\partial T}{\partial R}\right)^2\right]-s\left(\frac{\partial T}{\partial R}\right)^2\right\}\sqrt[3]{1+3\alpha_0T_0\left(1-\frac{T_0}{T}\right)}$$
$$=\frac{\rho_oc_E}{k_o}\dot{T}-6\frac{\mu_o}{k_o}\alpha_oT_oT\left[\frac{1}{R^2}-\frac{\left(1+3\alpha_0T_0\left(1-\frac{T_0}{T}\right)\right)^2R^4}{\left(\left(1+3\alpha_0T_0\right)R^3-9\alpha_0T_0\int_0^R\xi^2\frac{T_0}{T}d\xi\right)^2}\right]$$
$$\times\left[\frac{1+3\alpha_0T_0\left(1-\frac{T_0}{T}\right)}{\left(1+3\alpha_0T_0\right)R^3-9\alpha_0T_0\int_0^R\xi^2\frac{T_0}{T}d\xi}\right]^{1/3}\int_0^R\frac{\xi^2\dot{T}}{T^2}d\xi\,. \tag{86}$$

On the boundary of the ball, we consider a convection boundary condition, i.e.

$$k_o\left[1-s(T(R_o,t)-T_o)\right]\left.\frac{\partial T}{\partial R}\right|_{(R_o,t)}=h_o[T_o-T(R_o,t)]\,, \tag{87}$$

where h_o is the surface heat transfer coefficient at the boundary of the ball. We assume that h_o is constant and introduce the parameter $\gamma=h_oR_o/k_o$. As an initial temperature field, we consider a thermal inclusion of radius R_i, i.e.

$$T_{\text{init}}(R)=\begin{cases}T_i & R\leq R_i\,,\\ T_o & R>R_i\,.\end{cases} \tag{88}$$

In the scope of the classical theory of linearized elasticity, the thermal stresses are given by [7]:

$$\hat{\sigma}^{rr}=12\mu_o\alpha_o\left[\frac{1}{R_o^3}\int_0^{R_o}T(\xi,t)\xi^2d\xi-\frac{1}{R^3}\int_0^RT(\xi,t)\xi^2d\xi\right]\,, \tag{89a}$$
$$\hat{\sigma}^{\theta\theta}=\hat{\sigma}^{\phi\phi}=6\mu_o\alpha_o\left[\frac{2}{R_o^3}\int_0^{R_o}T(\xi,t)\xi^2d\xi+\frac{1}{R^3}\int_0^RT(\xi,t)\xi^2d\xi-T(R,t)\right]\,. \tag{89b}$$

In the classical linearized elasticity literature, the coupling term is neglected and the linearized heat equation problem for the sphere reads

$$\begin{cases}\dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2\dfrac{\partial T}{\partial R}\right)=\dfrac{\rho_oc_E}{k_o}\dot{T}\,,\\ T(R,0)=T_{\text{init}}(R)\,,\\ \left.\dfrac{\partial T}{\partial R}\right|_{(R_o,t)}=\dfrac{\gamma}{R_o}[T_o-T(R_o,t)]\,.\end{cases} \tag{90}$$

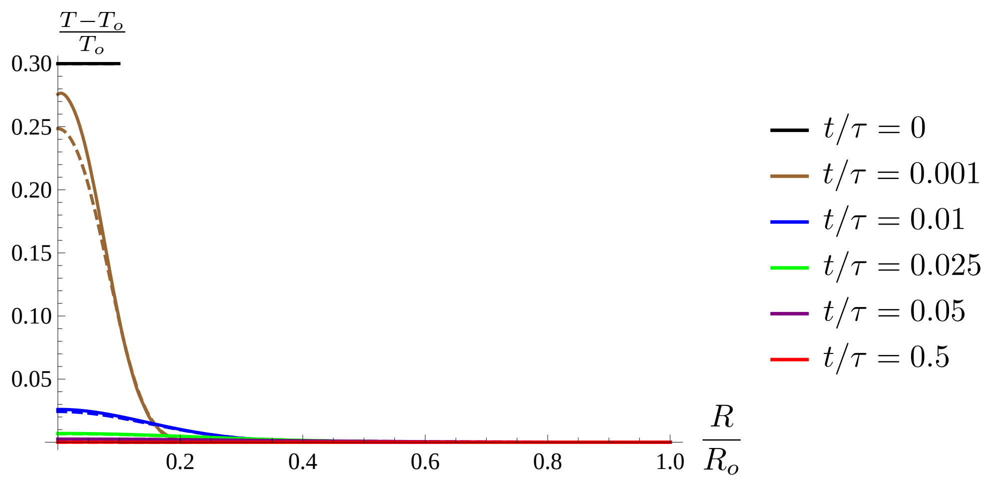


Figure 3: Temperature field in the sphere for $\frac{R_i}{R_o} = 0.1$, $\gamma = 1$, $\alpha_o T_o = 0.18$, $\mu_o/\rho_o c_E T_o = 0.001$ and $\delta_T = 30\%$ at different $\frac{t}{\tau}$ (Solid lines: nonlinear solution; dashed lines: linear solution).

The solution to (90) can be found analytically using a Fourier series expansion (see [10] for a detailed derivation.)

$$\begin{aligned} T(R,t) = 2\Delta_i T \sum_{n=1}^{\infty} \frac{\zeta_n^2 + (\gamma-1)^2}{\zeta_n^2 + \gamma(\gamma-1)} \sin\left(\zeta_n \frac{R}{R_o}\right) \left[\frac{1}{\zeta_n^2}\frac{R_o}{R}\sin\left(\zeta_n\frac{R_i}{R_o}\right)\right. \\ \left. - \frac{1}{\zeta_n}\frac{R_i}{R}\cos\left(\zeta_n\frac{R_i}{R_o}\right)\right] e^{-\frac{\zeta_n^2 t}{\tau}} + T_o\,, \end{aligned} \tag{91}$$

where ζ_n are the positive solutions of $\zeta \cot\zeta = 1-\gamma$.

We consider a rubber sphere of radius $R_o = 15\,\text{cm}$ for which the surface heat transfer coefficient for the rubber-air convection is $h_o = 10\,\text{W/m}^2.\text{K}$. We let $T_o = 300°\text{K}$ and $\delta_T = \frac{\Delta_i T}{T_o} = 30\%$ and assume the following typical values for rubber-like materials: $\rho_o = 10^3\,\text{kg/m}^3$, $c_E = 1800\,\text{J/kg.m}$, $k_o = 0.15\,\text{W/m.K}$, $s = 0,004\,\text{K}^{-1}$, $\alpha_o = 6\times10^{-4}\,\text{K}^{-1}$, and $\mu_o = 0.54\,\text{GPa}$. We numerically solve the initial/boundary value problem (86), (87), (88) for the temperature field $T(R,t)$ and show its evolution in Figure 3 by plotting $\frac{T(R,t)-T_o}{T_o}$ at different values of t/τ , where τ is a characteristic time defined as $\tau = \rho_o c_E R_o^2/k_o$. In Figures 4-5, we show the nonlinear thermal stresses (79). For comparison purposes, we also show the linearized solution (89) and (91) in Figures 3-5. We observe that the initial irregularities in the initial temperature and

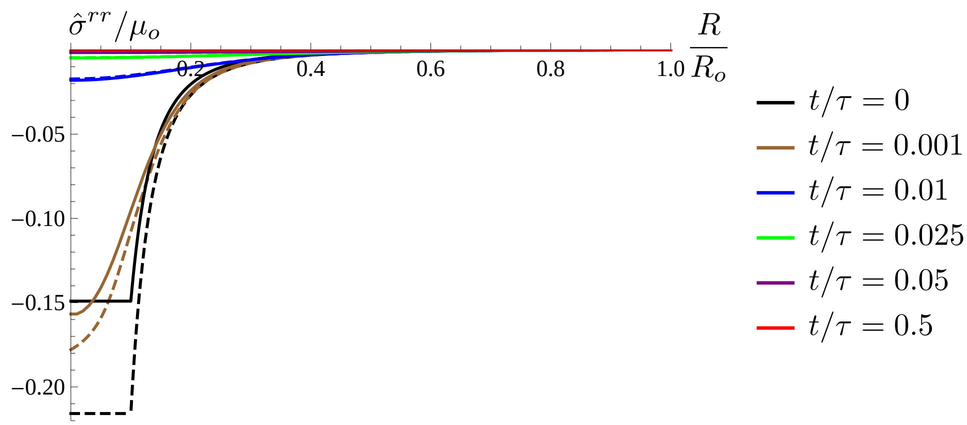


Figure 4: Radial stress field for $\frac{R_i}{R_o} = 0.1$, $\gamma = 1$, $\alpha_o T_o = 0.18$ and $\mu_o/\rho_o c_E T_o = 0.001$ at different $\frac{t}{\tau}$ (Solid lines: nonlinear solution; dashed lines: linear solution).

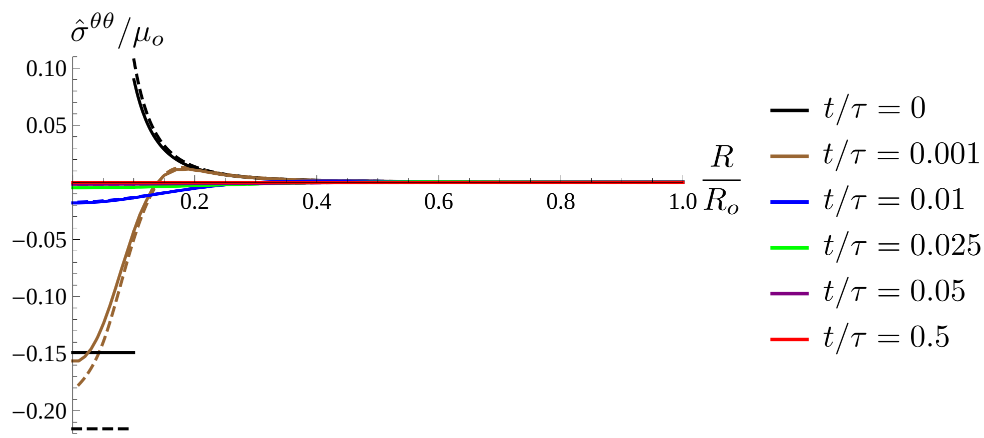


Figure 5: Circumferential stress field for $\frac{R_i}{R_o} = 0.1$, $\gamma = 1$, $\alpha_o T_o = 0.18$ and $\mu_o/\rho_o c_E T_o = 0.001$ at different $\frac{t}{\tau}$ (Solid lines: nonlinear solution; dashed lines: linear solution).

thermal stress fields are smoothed out; at large times both the temperature difference $T - T_o$ and thermal stress fields tend to zero. The nonlinear and linear solutions for the temperature field (Figure 3) show a similar trend but we observe a significant difference for thermal stress fields between the linear (89) and nonlinear (79) solutions (the maximum relative difference is of 38% inside the inclusion for $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$, see Figures 3-5). We also observe that in the nonlinear solution the maximum thermal stress does not necessarily correspond to $t = 0$, i.e. maximum stress occurs at a later time $t > 0$.

Remark 2.4.3. We observe that the coupling term in the nonlinear heat equation (86) is negligeable. In fact, if we neglect the coupling term in (86), the resulting solution is only affected in the order of 10^{-6}.

CHAPTER III

ANELASTICITY OF SHELLS

Many thin three-dimensional elastic bodies can be reduced to elastic shells: two-dimensional elastic bodies whose reference shape is not necessarily flat. These idealized objects are suitable models for many physical, engineering, and biological systems. Here, we formulate a general geometric theory of anelastic shells that describes both the evolution of the body shape, viewed as an orientable surface, as well as its intrinsic material properties such as its reference curvatures. In this geometric theory, anelastic eigenstrains are modeled using an evolving referential configuration for the shell. Geometric quantities attached to the surface, such as the first and second fundamental forms are obtained from the metric of the three-dimensional body and its evolution. The governing dynamical equations for the body are obtained from variational consideration by assuming that both fundamental forms on the material manifold are dynamical variables in a Lagrangian field theory. As an example, we apply these ideas to study morphoelastic shells, i.e., elastic shells that can remodel and grow in time. We find some stress-free growth fields of a planar sheet, and the residual stress field of a morphoelastic circular shell. Note that the results of this section have been previously reported in our published work [72].

3.1 Shell as a Hypersurface

Idealization of a thin body Let B be a three-dimensional thin body (i.e. its thickness is negligible compared to the other two dimensions) identified with an orientable three-dimensional Riemannian manifold $\mathcal{B}$ endowed with the metric $\bar{\boldsymbol{G}}$. Let H—the mid-surface of B—be identified with $(\mathcal{H}, \boldsymbol{G}, \boldsymbol{B})$, a two-dimensional Riemannian submanifold of $\left(\mathcal{B}, \bar{\boldsymbol{G}}\right)$ with first and second fundamental forms $\boldsymbol{G}$ and $\boldsymbol{B}$ (see

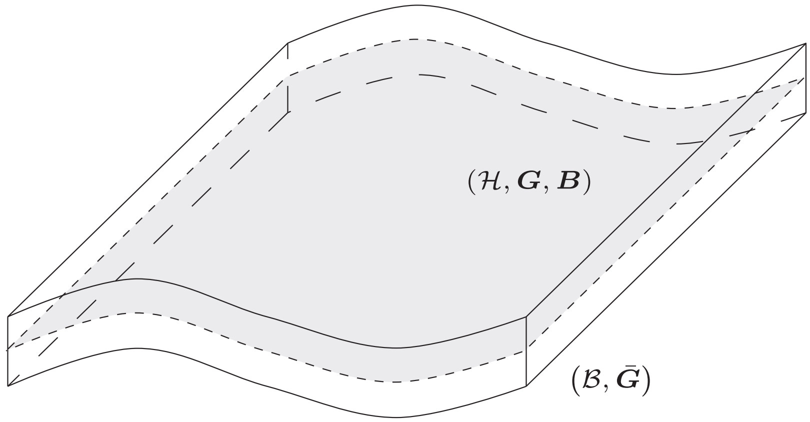


Figure 6: The mid-surface $(\mathcal{H}, \boldsymbol{G}, \boldsymbol{B})$ as a Riemannian submanifold of $\left(\mathcal{B}, \bar{\boldsymbol{G}}\right)$.

Figure 6). We assume in the following that $\mathcal{H}$ is an orientable hypersurface of $\mathcal{B}$. We show in the following that the natural isometric embedding of $\mathcal{H}$ in $\mathcal{B}$ induces independent in-plane and out-of-plane geometries for the hypersurface $\mathcal{H}$.

Let (X^1, X^2, X^3) be a local coordinate chart compatible with $\mathcal{H}$. In this coordinate chart, at any point $X \in \mathcal{B}$, the metric $\bar{\boldsymbol{G}}$ of $\mathcal{B}$ has the following representation:

$$\bar{\boldsymbol{G}}(X) = \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) & \bar{G}_{13}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) & \bar{G}_{23}(X) \\ \bar{G}_{13}(X) & \bar{G}_{23}(X) & \bar{G}_{33}(X) \end{pmatrix}.$$

If $X \in \mathcal{H}$, we have

$$\bar{\boldsymbol{G}}(X) = \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) & 0 \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the metric $\boldsymbol{G}$ of $\mathcal{H}$, referred to as the first fundamental form, has the following representation

$$\boldsymbol{G}(X) = \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) \end{pmatrix}, \ \forall X \in \mathcal{H}, \tag{92}$$

and the second fundamental form of $\mathcal{H}$ has the following components

$$B_{AB}(X) = \bar{\Gamma}^3{}_{AB}(X)\,, \ A, B = 1, 2\,, \ \forall X \in \mathcal{H}\,,$$

where $\bar{\Gamma}^C{}_{AB} = \frac{1}{2}\sum_K \bar{G}^{CK}\left(\partial_A \bar{G}_{KB} + \partial_B \bar{G}_{KA} - \partial_K \bar{G}_{AB}\right)$ is the Christoffel symbol of the Levi-Civita connection of $\left(\mathcal{B}, \bar{\boldsymbol{G}}\right)$. Therefore

$$B_{AB}(X) = -\frac{1}{2}\left.\frac{\partial \bar{G}_{AB}}{\partial X^3}\right|_{\mathcal{H}}(X)\,,\ A, B = 1, 2\,,\ \forall X \in \mathcal{H}\,, \tag{93}$$

where $\bar{G}_{AB}$ should be thought of as a function on the coordinate curve X^3 and $\frac{\partial \bar{G}_{AB}}{\partial X^3}|_{\mathcal{H}}$ is evaluated at the point where the curve X^3 meets the hypersurface $\mathcal{H}$. Since $\bar{G}_{AB}$ and $\frac{\partial \bar{G}_{AB}}{\partial X^3}$ can be prescribed independently, equations (92) and (93) demonstrate that independent first and second fundamental forms $\mathbf{G}$ and $\mathbf{B}$ of the hypersurface $\mathcal{H}$ can be obtained from the metric $\bar{\boldsymbol{G}}$ of the embedding space $\mathcal{B}$. Therefore, we only need to specify the components $\bar{G}_{AB}$ for $A, B = 1, 2$ to characterize the geometry of $\mathcal{H}$. We introduce the following notation

$$\bar{\boldsymbol{G}}_{\mathcal{H}}(X) := \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) \end{pmatrix}\,,\ \forall X \in \mathcal{B}\,. \tag{94}$$

Remark 3.1.1. In the local coordinate chart (X^1, X^2, X^3), note that the components $\bar{G}_{A3}\,,\ A = 1, 2, 3$ of the metric $\bar{\boldsymbol{G}}$ do not affect the geometry of $\mathcal{H}$. Indeed, from equations (92) and (93), the geometry of $\mathcal{H}$ depends only on the restriction of the metric $\bar{\boldsymbol{G}}$ to $\mathcal{H}$ (i.e., $\bar{G}_{AB}|_{\mathcal{H}}\,,\ A, B = 1, 2$) and its first order derivative along the normal to $\mathcal{H}$ (i.e., $\frac{\partial \bar{G}_{AB}}{\partial X^3}|_{\mathcal{H}}\,,\ A, B = 1, 2$). Therefore, higher order variations of $\bar{\boldsymbol{G}}_{\mathcal{H}}$ along the thickness of B are not captured by the geometry of $\mathcal{H}$. As an example, let $\mathcal{B}$ be a thin body in $\mathbb{R}^3$ with the coordinate chart (X^1, X^2, X^3) such that the hypersurface $X^3 = 0$ contains the mid-surface $\mathcal{H}$. Two different metrics for $\mathcal{B}$ such that $\bar{G}^1_{AB}(X) = e^{X^3}\delta_{AB}\,,$ and $\bar{G}^2_{AB}(X) = (1 + X^3)\,\delta_{AB}\,,\ \forall X \in \mathcal{B}\,,\ A, B = 1, 2\,,$ correspond to the same geometry for $\mathcal{H}$ given by $G_{AB} = \delta_{AB}$ and $B_{AB} = -\frac{1}{2}\delta_{AB}\,,\ A, B = 1, 2\,.$ Also, if we consider an evolving metric such that $\bar{G}_{AB}(X, t) = (1 + (X^3)^2 f(t))\,\delta_{AB}\,,\ \forall X \in \mathcal{B}\,,\ A, B = 1, 2\,,$ where f is a given function of time, we find that the geometry of $\mathcal{H}$ does not capture this evolution as it remains unchanged both in-plane and out-of-plane ($\boldsymbol{G} = \boldsymbol{\delta}$ and $\boldsymbol{B} = \mathbf{0}$).

Evolving material geometry and anelasticity in a shell In order to model anelasticity in shells, following the discussion in 2.2, we assume an evolving metric for the material manifold $(\mathcal{B}, \bar{\boldsymbol{G}})$, i.e., we leave the manifold $\mathcal{B}$ fixed and endow it with an evolving metric[1] $\bar{\boldsymbol{G}}$, i.e., $\bar{\boldsymbol{G}} = \bar{\boldsymbol{G}}(X,t)$, such that at $t=0$, we have $\bar{\boldsymbol{G}}(X,0) = \bar{\boldsymbol{G}}^0(X)$ the metric of a natural stress-free configuration of $\mathcal{B}$. In this work, however, we are interested in anelasticity in thin bodies, and hence, we consider the manifold $(\mathcal{B}, \bar{\boldsymbol{G}})$ with an evolving metric such that, in a local coordinate chart (X^1, X^2, X^3) compatible with $\mathcal{H}$, only $\bar{\boldsymbol{G}}_{\mathcal{H}}$ is evolving.[2] Now, we leave the mid-surface manifold $\mathcal{H}$ fixed and let its evolving first and second fundamental forms $\boldsymbol{G}$ and $\boldsymbol{B}$ be induced from its natural isometric embedding in $\mathcal{B}$. Therefore, in the local coordinate chart (X^1, X^2, X^3), the metric reads

$$\boldsymbol{G}(X,t) = \begin{pmatrix} \bar{G}_{11}(X,t) & \bar{G}_{12}(X,t) \\ \bar{G}_{12}(X,t) & \bar{G}_{22}(X,t) \end{pmatrix}, \ \forall X \in \mathcal{H}, \tag{95}$$

and the second fundamental form of $\mathcal{H}$ is written as

$$B_{AB}(X,t) = -\frac{1}{2} \left.\frac{\partial \bar{G}_{AB}}{\partial X^3}\right|_{\mathcal{H}} (X,t), \ A,B = 1,2, \ \forall X \in \mathcal{H}. \tag{96}$$

We will discuss, in §3.3, the governing equations for the evolution of the first and the second fundamental forms of shells and see how the evolving geometry of the material manifold is coupled with its current state of stress. Note that the evolving fundamental forms $\boldsymbol{G}$ and $\boldsymbol{B}$ are compatible in the manifold $(\mathcal{B}, \bar{\boldsymbol{G}})$, i.e., they satisfy the Gauss and Codazzi-Mainardi equations (253) and (254).

To illustrate the evolving geometry of a surface, we consider a flat thin body that can be represented by a planar surface $(\mathcal{H}, \boldsymbol{G})$ in $(\mathbb{R}^3, \bar{\boldsymbol{G}})$. Let (X^1, X^2, X^3) be the standard coordinate chart for $\mathbb{R}^3$ such that the hyperplane $X^3 = 0$ contains the surface $\mathcal{H}$. If we assume that the body undergoes an evolution that is uniform through its thickness, then we can model it by an evolving metric $\bar{\boldsymbol{G}}_{\mathcal{H}}$ such that

[1] Other examples of evolving material metrics in mechanics have been introduced in [62, 90, 91, 92, 88, 94, 27].

[2] cf. (94) where the notation $\bar{\boldsymbol{G}}_{\mathcal{H}}$ was introduced.

$\bar{G}_{AB}(X,t) = \bar{G}_{AB}(X^1, X^2, t)$, $\forall X \in \mathcal{B}$, $A, B = 1, 2$ (i.e., $\bar{G}_{AB}$'s do not depend on X^3), then, we obtain an evolving geometry for $\mathcal{H}$ with an evolving metric $G_{AB}(X,t) = \bar{G}_{AB}(X,t)$, $\forall X \in \mathcal{H}$ and a vanishing second fundamental form. As an example $\bar{G}_{AB}(X^1, X^2, t) = f(t)\delta_{AB}$, $A, B = 1, 2$, for some function f of time, models a uniform in-plane evolution with no out-of-plane geometry change (i.e., a vanishing second fundamental form). However, if we assume that the body undergoes an evolution that is not uniform through its thickness we obtain an evolving geometry for $\mathcal{H}$ such that the second fundamental form evolves with time: $B_{AB}(X,t) = -\frac{1}{2}\frac{\partial \bar{G}_{AB}}{\partial X^3}|_{\mathcal{H}}(X,t)$. As an example, we let $\bar{G}_{AB}(X,t) = f(X^3,t)\delta_{AB}$, $A, B = 1, 2$, for some function f of time and X^3 such that $f(0,t) = 1$, $\frac{\partial f}{\partial X^3}(0,0) = 0$, and $\frac{\partial f}{\partial X^3}(0,t) \neq 0$ for $t \neq 0$ (e.g. $f(X^3,t) = e^{X^3 t}$). Then we have an evolving geometry for $\mathcal{H}$ such that the metric of $\mathcal{H}$ remains unchanged: $G_{AB} = \delta_{AB}$ while its out-of-plane geometry evolves with time: $B_{AB} = -\frac{1}{2}\frac{\partial f}{\partial X^3}(0,t)\delta_{AB}$.

Remark 3.1.2. Given the fact (discussed in Remark 3.1.1) that the geometry of $\mathcal{H}$ depends only on the restriction of the metric $\bar{\boldsymbol{G}}$ to $\mathcal{H}$ and its first order derivative along the normal to $\mathcal{H}$, we are bound to model a restrictive class of material evolutions. We assume that the evolving material manifold $(\mathcal{H}, \boldsymbol{G}(.,t))$ at time t is diffeomorphic to the reference manifold $(\mathcal{H}, \boldsymbol{G}^0)$ at time $t = 0$, so that this diffeomorphism can be extended to a neighborhood of $\mathcal{H}$ in $\mathcal{B}$ in such a way that the push-forward of the reference normal vector field $\boldsymbol{N}^0$ of $\mathcal{H}$ in $(\mathcal{B}, \bar{\boldsymbol{G}}^0)$ is precisely the evolving normal vector field $\boldsymbol{N}$ of $\mathcal{H}$ in $(\mathcal{B}, \bar{\boldsymbol{G}}^0)$. Note that this implies that during the material evolution of the shell, at any point of $\mathcal{H}$, the normal to $\mathcal{H}$ remains normal.

We can write the evolving metric $\bar{\boldsymbol{G}}_{\mathcal{H}}$ in the form

$$\bar{\boldsymbol{G}}_{\mathcal{H}}(X,t) = \bar{\boldsymbol{G}}^0_{\mathcal{H}}(X) e^{2\bar{\boldsymbol{\omega}}(X,t)}, \quad \forall X \in \mathcal{B},$$

where $\bar{\boldsymbol{\omega}}$ is a smooth $\binom{1}{1}$-rank tensor characterizing growth of the thin body $\mathcal{B}$ such that $\bar{\boldsymbol{\omega}}(X,0) = \mathbf{0}$. Following (95), the evolving first fundamental form of $\mathcal{H}$ is given by

$$\boldsymbol{G}(X,t)=\boldsymbol{G}^0(X)e^{2\boldsymbol{\omega}(X,t)}\,,\quad \forall X\in\mathcal{H}\,,$$

where $\boldsymbol{G}^0=\bar{\boldsymbol{G}}^0_{\mathcal{H}}\big|_{\mathcal{H}}$ and $\omega=\bar{\omega}|_{\mathcal{H}}$. Following (96), for $X\in\mathcal{H}$, the evolving second fundamental form of $\mathcal{H}$ is given by

$$\begin{aligned}\boldsymbol{B}(X,t)&=-\frac{1}{2}\frac{\partial\bar{\boldsymbol{G}}_{\mathcal{H}}}{\partial X^3}(X,t)\\&=-\frac{1}{2}\frac{\partial}{\partial X^3}\left[\bar{\boldsymbol{G}}^0_{\mathcal{H}}e^{2\bar{\boldsymbol{\omega}}}\right](X,t)\\&=-\frac{1}{2}\frac{\partial\bar{\boldsymbol{G}}^0_{\mathcal{H}}}{\partial X^3}(X)e^{2\bar{\boldsymbol{\omega}}(X,t)}-\bar{\boldsymbol{G}}^0_{\mathcal{H}}(X)\frac{\partial\bar{\boldsymbol{\omega}}}{\partial X^3}(X,t)e^{2\bar{\boldsymbol{\omega}}(X,t)}\,.\end{aligned}$$

For $X\in\mathcal{H}$, we introduce the following notations:

$$\boldsymbol{B}^0(X):=-\frac{1}{2}\frac{\partial\bar{\boldsymbol{G}}^0_{\mathcal{H}}}{\partial X^3}(X)\,,\quad \omega(X,t):=\bar{\omega}(X,t)\ ,\ \text{and}\quad \boldsymbol{K}(X,t)=\frac{\partial\bar{\boldsymbol{\omega}}}{\partial X^3}(X,t)\,,$$

and hence we write the evolving first and second fundamental forms of $\mathcal{H}$ as

$$\boldsymbol{G}(X,t)=\boldsymbol{G}^0(X)e^{2\boldsymbol{\omega}(X,t)}\,,\quad \boldsymbol{B}(X,t)=\boldsymbol{B}^0(X)e^{2\boldsymbol{\omega}(X,t)}-\boldsymbol{G}^0(X)\boldsymbol{K}(X,t)e^{2\boldsymbol{\omega}(X,t)}\,,\tag{97}$$

such that $\boldsymbol{\omega}(X,0)=\mathbf{0}$ and $\boldsymbol{K}(X,0)=\mathbf{0}$, so that at $t=0$, $\boldsymbol{G}(t=0)=\boldsymbol{G}^0$ and $\boldsymbol{B}(t=0)=\boldsymbol{B}^0$. For an isotropic evolution, we have $\omega_1=\omega_2=\omega$ and $K_1=K_2=K$, recalling that for $A=\{1,2\}$, $K_A=\frac{\partial\bar{\omega}_A}{\partial X^3}$, hence, we obtain

$$\boldsymbol{G}(X,t)=\boldsymbol{G}^0(X)e^{2\omega(X,t)}\,,\quad \boldsymbol{B}(X,t)=\boldsymbol{B}^0(X)e^{2\omega(X,t)}-K(X,t)\boldsymbol{G}^0(X)e^{2\omega(X,t)}\,,\tag{98}$$

such that $\omega(X,0)=0$ and $K(X,0)=0$.

Remark 3.1.3. The Riemannian surface form—i.e., the volume form of the surface $\mathcal{H}$—associated with the metric $\boldsymbol{G}$ is written as

$$dS(X,\boldsymbol{G})=\sqrt{\det\boldsymbol{G}}\,dX^1\wedge dX^2=\sqrt{\det\boldsymbol{G}^0}\,e^{\mathrm{tr}(\boldsymbol{\omega}(X,T))}dX^1\wedge dX^2=e^{\mathrm{tr}(\boldsymbol{\omega}(X,t))}dS_0(X)\,,$$

where dS_0 is the Riemannian surface form associated with the metric $\boldsymbol{G}^0$. Using the identity $\det e^{\boldsymbol{A}}=e^{\mathrm{tr}(\boldsymbol{A})}$, the rate of change of the volume element due to the evolving metric is given by

$$\frac{d}{dt}dS(X,\boldsymbol{G})=\frac{d}{dt}\left[\mathrm{tr}(\boldsymbol{\omega}(X,t))\right]dS(X,\boldsymbol{G})\,.\tag{99}$$

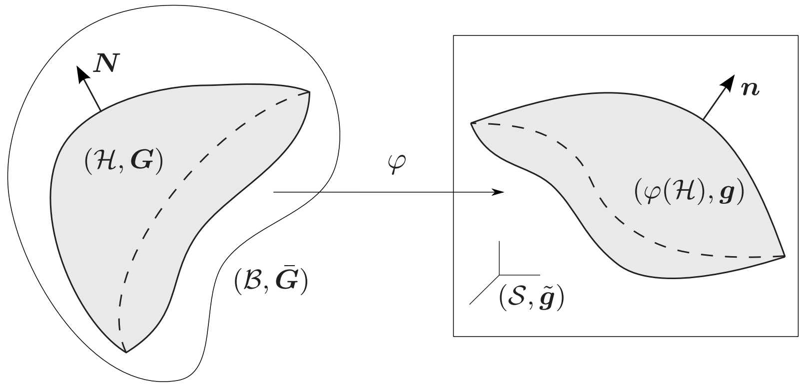


Figure 7: A configuration $\varphi : \mathcal{H} \to \mathcal{S}$ of a Riemannian surface $(\mathcal{H}, \boldsymbol{G})$ in the ambient space $(\mathcal{S}, \tilde{\boldsymbol{g}})$. The vector fields $\boldsymbol{N}$ and $\boldsymbol{n}$ are the unit normal vector fields of $\mathcal{H}$ and $\varphi(\mathcal{H})$, respectively.

Alternatively, by using the identity $\frac{d}{dt}\left[\det \boldsymbol{A}(t)\right] = \det \boldsymbol{A}(t) \operatorname{tr}\left[\boldsymbol{A}^{-1}(t) \frac{d}{dt}\boldsymbol{A}(t)\right]$, we find that

$$\frac{d}{dt} dS(X, \boldsymbol{G}) = \frac{1}{2} \operatorname{tr}\left(\frac{d\boldsymbol{G}}{dt}\right) dS(X, \boldsymbol{G}) .$$

3.2 Kinematics of Shells

Let the ambient space be $\mathcal{S} = \mathbb{R}^3$ endowed with the standard Euclidean metric $\tilde{\boldsymbol{g}}$. Recall that the Riemannian surface $(\mathcal{H}, \boldsymbol{G}, \boldsymbol{B})$ is an orientable two-dimensional Riemannian submanifold of $\left(\mathcal{B}, \bar{\boldsymbol{G}}\right)$. A configuration of $\mathcal{H}$ in $\mathcal{S}$ is a smooth embedding $\varphi : \mathcal{H} \to \mathcal{S}$. We denote the set of all configurations of $\mathcal{H}$ in $\mathcal{S}$ by $\mathcal{C}$. As shown in Figure 7, the Riemannian manifold $(\varphi(\mathcal{H}), \boldsymbol{g}, \boldsymbol{\beta})$, where $\boldsymbol{g} := \tilde{\boldsymbol{g}}|_{\varphi(\mathcal{H})}$ and $\boldsymbol{\beta} \in \Gamma(S^2 T^* \varphi(\mathcal{H}))$ the second fundamental form of $\varphi(\mathcal{H})$, is a hypersurface in $\mathcal{S}$. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$, respectively. Let $\boldsymbol{n} \in \mathfrak{X}(\varphi(\mathcal{H}))^{\perp}$ be the smooth unit normal vector field of $\varphi(\mathcal{H})$ and $\boldsymbol{\mathcal{R}} \in \Gamma(S^4 T^* \varphi(\mathcal{H}))$ be the Riemannian curvature of the surface $\varphi(\mathcal{H})$. Since the ambient space $\mathcal{S} = \mathbb{R}^3$ is flat, the Gauss and Codazzi-Mainardi equations for the Riemannian manifold $(\varphi(\mathcal{H}), \boldsymbol{g})$

read

$$\boldsymbol{\mathcal{R}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) = \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{w}) - \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{z})\,, \tag{100a}$$

$$\left(\nabla_{\boldsymbol{x}}\boldsymbol{\beta}\right)(\boldsymbol{y}, \boldsymbol{z}) = \left(\nabla_{\boldsymbol{y}}\boldsymbol{\beta}\right)(\boldsymbol{z}, \boldsymbol{z})\,. \tag{100b}$$

We denote objects and indices by uppercase characters in the material manifold (e.g., $X \in \mathcal{H}$ for a material point) and by lowercase characters in the spatial manifold (e.g., $x \in \varphi(\mathcal{H})$ for a spatial point). In the remainder of the paper, unless stated otherwise, all indices (material and spatial) take values in the range $1, 2$. We adopt the standard Einstein convention of summation over repeated indices.

Strain measures We define the deformation gradient $\boldsymbol{F}$ as the tangent map of $\varphi : \mathcal{H} \to \varphi\left(\mathcal{H}\right)$, i.e., $\boldsymbol{F}(X) := T_X\varphi : T_X\mathcal{H} \mapsto T_{\varphi(X)}\varphi\left(\mathcal{H}\right)$. The right Cauchy-Green deformation tensor $\boldsymbol{C} \in \Gamma(S^2T^*\mathcal{H})$ is defined as the pull back of the spatial metric [50], $\boldsymbol{C}(X) := \varphi^*\boldsymbol{g}(X) : T_X\mathcal{H} \mapsto T_X\mathcal{H}$, i.e., $\boldsymbol{C}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\varphi_*\boldsymbol{X}, \varphi_*\boldsymbol{Y})\,, \forall \boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\mathcal{H})$. In components, $C_{AB} = F^a{}_A F^b{}_B g_{ab}$. The Jacobian of the motion J relates the material and spatial Riemannian surface forms $dS(X, \boldsymbol{G})$ and $ds(\varphi(X), \boldsymbol{g})$ by

$$\varphi^* ds = J dS\,.$$

It can be shown that [50]

$$J = \sqrt{\frac{\det \varphi^*\boldsymbol{g}}{\det \boldsymbol{G}}}\,. \tag{101}$$

The material strain tensor $\boldsymbol{E} \in \Gamma(S^2T^*\mathcal{H})$ is given by $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{G}\right)$. The spatial strain tensor $\boldsymbol{e} \in \Gamma(S^2T^*\varphi(\mathcal{H}))$ is defined as $\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{g} - \boldsymbol{c}\right)$, where $\boldsymbol{c} = \varphi_*\boldsymbol{G}$. Note that $\boldsymbol{e} = \varphi_*\boldsymbol{E}$. The material and spatial strain tensors are intrinsic in the sense that they are determined by the metrics of the reference and the final configurations of the surface. We introduce extrinsic strain tensors for configurations of surfaces that depend on the second fundamental form as follows. The extrinsic deformation tensor $\boldsymbol{\Theta} \in \Gamma(S^2T^*\mathcal{H})$ is defined as the pull back of the spatial second fundamental form

$$\boldsymbol{\Theta} := \varphi^*\boldsymbol{\beta}\,.$$

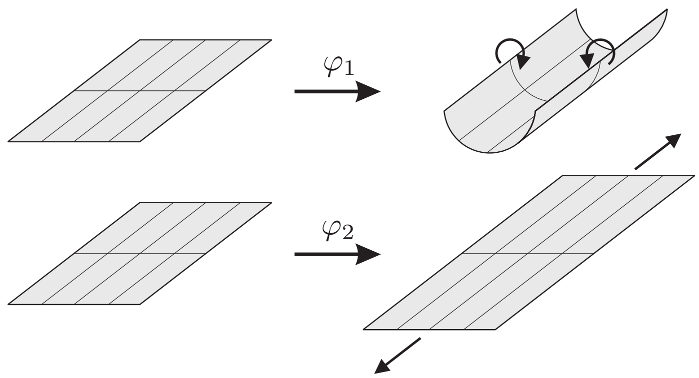


Figure 8: Strains of two different configurations of a sheet: (i) The configuration φ_1 maps the sheet to a section of a cylinder with $\boldsymbol{E}=\mathbf{0}$, and $\boldsymbol{H}\neq\mathbf{0}$, (ii) the configuration φ_2 is an in-plane extension of the sheet with $\boldsymbol{E}\neq\mathbf{0}$, and $\boldsymbol{H}=\mathbf{0}$.

In components, $\Theta_{AB} = F^a{}_A F^b{}_B \beta_{ab}$. We define the extrinsic material strain tensor as $\boldsymbol{H} := \frac{1}{2}(\boldsymbol{\Theta}-\boldsymbol{B})$, and the extrinsic spatial strain tensor as $\boldsymbol{\eta} := \frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{\theta})$, where $\boldsymbol{\theta} := \varphi_*\boldsymbol{B}$. Note that $\boldsymbol{\eta} = \varphi_*\boldsymbol{H}$. As an example, consider two different configurations of a sheet shown in Figure 8. The configuration φ_1 is an isometry between the sheet and a section of a cylinder, and therefore, $\boldsymbol{E}=\mathbf{0}$. However, note that since the out-of-plane geometry has changed, we have $\boldsymbol{H}\neq\mathbf{0}$. On the other hand, φ_2 is an in-plane deformation of the sheet with $\boldsymbol{E}\neq\mathbf{0}$, and $\boldsymbol{H}=\mathbf{0}$.

Compatibility equations of shells The pull-back of the Gauss and the Codazzi equations (100) of the surface $(\varphi(\mathcal{H}),\boldsymbol{g})$ by φ read [3]

$$\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = \boldsymbol{\Theta}(\boldsymbol{X},\boldsymbol{Z})\boldsymbol{\Theta}(\boldsymbol{Y},\boldsymbol{W}) - \boldsymbol{\Theta}(\boldsymbol{X},\boldsymbol{W})\boldsymbol{\Theta}(\boldsymbol{Y},\boldsymbol{Z}), \tag{102a}$$

$$\left(\nabla^{\boldsymbol{C}}_{\boldsymbol{X}}\boldsymbol{\Theta}\right)(\boldsymbol{Y},\boldsymbol{Z}) = \left(\nabla^{\boldsymbol{C}}_{\boldsymbol{Y}}\boldsymbol{\Theta}\right)(\boldsymbol{X},\boldsymbol{Z}), \tag{102b}$$

where $\nabla^{\boldsymbol{C}}$ and $\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}$ are, respectively, the Levi-Civita connection and the Riemannian curvature of the Riemannian manifold $(\mathcal{H},\boldsymbol{C})$. Given a metric $\boldsymbol{C}\in\Gamma(S^2T^*\mathcal{H})$ and a symmetric tensor $\boldsymbol{\Theta}\in\Gamma(S^2T^*\mathcal{H})$, the relations (102) express the compatibility equations for these tensors when $\mathcal{H}$ is simply-connected, i.e., they are the necessary and locally sufficient conditions for the existence of a configuration of $\mathcal{H}$ with the given deformation tensors that is unique up to isometries of $\mathcal{S} = \mathbb{R}^3$ when $\mathcal{H}$ is simply-connected [36, 3]. Hence, we observe that if φ_1 and φ_2 are different configurations of the surface $\mathcal{H} \subset \mathbb{R}^3$ with the same deformation tensors, then $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are rigid body motions of $\mathcal{H}$ in $\mathbb{R}^3$. Note that similar to (255), given the symmetries of the curvature tensor $\boldsymbol{\mathcal{R}}^{\boldsymbol{C}}$ and the extrinsic deformation tensor, the compatibility equations reduce in components to

$$
\begin{aligned}
&\mathcal{R}^C_{1212} = \Theta_{11}\Theta_{22} - \Theta_{12}\Theta_{12}\,, \\
&\Theta_{11||2} = \Theta_{12||1}\,, \\
&\Theta_{22||1} = \Theta_{12||2}\,,
\end{aligned}
$$

where we denote by a double stroke $||$ the covariant derivative corresponding to the Levi-Civita connection of $(\mathcal{H}, \boldsymbol{C})$.

Stress-free shell evolution Given a thin body $\mathcal{B}$ and its idealization, the mid-surface $\mathcal{H}$, we want to find those evolution fields that leave the shell "stress-free", by which we mean both stress and couple-stress free. As introduced earlier, given the smooth embedding φ of the surface $(\mathcal{H}, \boldsymbol{G}, \boldsymbol{B})$ into the Euclidean space $\mathcal{S}$ to form a surface $(\varphi(\mathcal{H}), \boldsymbol{g}, \boldsymbol{\beta})$, the tensors $2\boldsymbol{E} = \varphi^*\boldsymbol{g} - \boldsymbol{G}$ and $2\boldsymbol{H} = \varphi^*\boldsymbol{\beta} - \boldsymbol{B}$, respectively, provide measures of in-plane and out-of-plane strains. Therefore, the surface is stress-free when these two measures are identically zero, i.e., $\varphi^*\boldsymbol{g} = \boldsymbol{G}$ and $\varphi^*\boldsymbol{\beta} = \boldsymbol{B}$. Noting that $\boldsymbol{C} = \varphi^*\boldsymbol{g}$ and $\boldsymbol{\Theta} := \varphi^*\boldsymbol{\beta}$ are uniquely specified by (102) when $\mathcal{H}$ is simply-connected, it follows that a simply-connected shell $\mathcal{H}$ is stress-free if and only if $\boldsymbol{G}$ and $\boldsymbol{B}$ are specified by (102), i.e., we have

$$\boldsymbol{\mathcal{R}}^{\mathcal{H}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = \boldsymbol{B}(\boldsymbol{X},\boldsymbol{W})\,\boldsymbol{B}(\boldsymbol{Y},\boldsymbol{Z}) - \boldsymbol{B}(\boldsymbol{X},\boldsymbol{Z})\,\boldsymbol{B}(\boldsymbol{Y},\boldsymbol{W})\,, \tag{104a}$$

$$\left(\nabla^{\mathcal{H}}_{\boldsymbol{Y}}\boldsymbol{B}\right)(\boldsymbol{X},\boldsymbol{Z}) = \left(\nabla^{\mathcal{H}}_{\boldsymbol{X}}\boldsymbol{B}\right)(\boldsymbol{Y},\boldsymbol{Z})\,. \tag{104b}$$

These are precisely the necessary and sufficient conditions for the simply-connected surface $(\mathcal{H}, \boldsymbol{G}, \boldsymbol{B})$ to be isometrically embeddable in $\mathbb{R}^3$. In components, equations

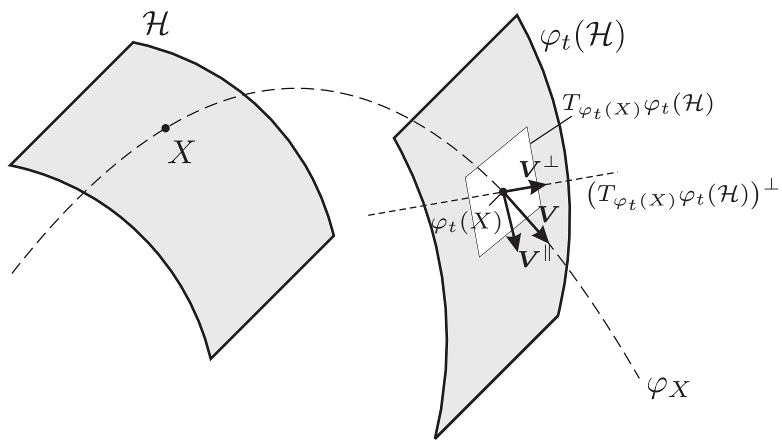


Figure 9: The decomposition of the material velocity $\boldsymbol{V} = \boldsymbol{V}^\parallel + \boldsymbol{V}^\perp$. The component $\boldsymbol{V}^\parallel(X,t)$ is an element of $T_{\varphi(X)}\varphi(\mathcal{H})$ and $\boldsymbol{V}^\perp(X,t)$ is normal to $T_{\varphi(X)}\varphi(\mathcal{H})$.

(104) reduce to

$$\begin{aligned} \mathcal{R}^{\mathcal{H}}_{1221} &= B_{11}B_{22} - B_{12}B_{12}\,, \\ B_{11|2} &= B_{12|1}\,, \\ B_{22|1} &= B_{12|2}\,. \end{aligned}$$

Velocity and acceleration We define a motion to be a smooth curve $t \mapsto \varphi_t \in \mathcal{C}$, i.e., $\varphi_t : \mathcal{H} \to \mathcal{S}$ and denote $\varphi(.,t) := \varphi_t(.)$ and $\varphi_X(.) := \varphi(X,.)$. At time t, the surface $\varphi_t(\mathcal{H})$ has the metric $\boldsymbol{g} := \tilde{\boldsymbol{g}}|_{\varphi_t(\mathcal{H})}$, the Levi-Civita connection ∇, the unit normal vector field $\boldsymbol{n} \in \mathfrak{X}(\varphi_t(\mathcal{H}))^\perp$, and the second fundamental form $\boldsymbol{\beta} \in \Gamma(S^2T^*\varphi_t(\mathcal{H}))$. The material velocity is the mapping $\boldsymbol{V} : \mathcal{H} \times \mathbb{R} \to T\mathcal{S}\,, (X,t) \mapsto \boldsymbol{V}(X,t) := T_t\varphi_X\left[\partial_t\right]$, $\forall X \in \mathcal{H}$. We denote for each $X \in \mathcal{H}$, $\boldsymbol{V}_X := \boldsymbol{V}(X,\cdot)$ the vector field along the curve φ_X, i.e., $\boldsymbol{V}_X \in \mathfrak{X}(\varphi_X)$. Using the decomposition of $T\mathcal{S}$, the material velocity can be decomposed as $\boldsymbol{V}_X(t) = \boldsymbol{V}^\parallel_X(t) + \boldsymbol{V}^\perp_X(t)$, where $\boldsymbol{V}^\parallel_X(t) \in T_{\varphi_t(X)}\varphi_t(\mathcal{H})$ and $\boldsymbol{V}^\perp_X(t) \in \left(T_{\varphi_t(X)}\varphi_t(\mathcal{H})\right)^\perp$, i.e., $\boldsymbol{V}^\parallel$ is parallel to $\varphi_t(\mathcal{H})$ and $\boldsymbol{V}^\perp$ is normal to $\varphi_t(\mathcal{H})$, see Figure 9. The spatial velocity at a fixed time t is a vector field along $\varphi_t(\mathcal{H})$ defined as $\boldsymbol{v}(x,t) := \boldsymbol{V}\left(\varphi_t^{-1}(X),t\right)$, where $x = \varphi_t^{-1}(X) \in \varphi_t(\mathcal{H}) \subset \mathcal{S}$. Note that even though for a fixed t, the mapping $\varphi_t : \mathcal{H} \to \mathcal{S}$ is a smooth embedding, the mapping $\varphi : \mathcal{H} \times \mathbb{R} \to \mathcal{S}$ is, in general, not even an immersion. In fact, it can be seen that $T_{(X,t)}\varphi$ is not necessarily injective. In $\{X^A\}$ and $\{x^a\}$, some local coordinate charts for $\mathcal{H}$ and $\mathcal{S}$, respectively, $T_{(X,t)}\varphi$ reads as follows

$$T_{(X,t)}\varphi = \begin{pmatrix} \frac{\partial\varphi^1}{\partial X^1} & \frac{\partial\varphi^1}{\partial X^2} & \frac{\partial\varphi^1}{\partial t} \\ \frac{\partial\varphi^2}{\partial X^1} & \frac{\partial\varphi^2}{\partial X^2} & \frac{\partial\varphi^2}{\partial t} \\ \frac{\partial\varphi^3}{\partial X^1} & \frac{\partial\varphi^3}{\partial X^2} & \frac{\partial\varphi^3}{\partial t} \end{pmatrix}.$$

Now, if $\boldsymbol{V}(X,t) = \mathbf{0}$ (i.e., $\partial\varphi^a/\partial t = 0$ for $a = 1,2,3$), or φ is an in-plane motion (i.e., in some coordinate chart for $\mathcal{S}$ such that $\partial_3 = \boldsymbol{n}$ on $\varphi_t(\mathcal{H})$ we have $\varphi^3 = 0$), $T_{(X,t)}\varphi$ is clearly not injective. However, if $T_{(X,t)}\varphi$ is injective, the implicit function theorem implies that φ is a local diffeomorphism at (X,t), and one can construct a local vector field $\boldsymbol{\mathcal{V}}$ on $\mathcal{S}$ in a neighborhood of $\varphi(X,t)$ such that $\boldsymbol{\mathcal{V}}(\varphi(X,t)) = \boldsymbol{V}(X,t) = \boldsymbol{v}(\varphi(X,t),t)$. Hence, the material acceleration can be in this case unambiguously defined as

$$\boldsymbol{A}(X,t) = D_{\varphi_X}\boldsymbol{V}_X := \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}(\varphi(X,t)),$$

where D_{φ_X} is the covariant derivative along φ_X. Using the decomposition of the material velocity into parallel and normal components $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}^{\parallel} + \boldsymbol{\mathcal{V}}^{\perp}$ and assuming that φ is a local diffeomorphism at (X,t), one can write

$$\boldsymbol{A}(X,t) = \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}(\boldsymbol{\mathcal{V}}^{\parallel} + \boldsymbol{\mathcal{V}}^{\perp}) = \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\parallel} + \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}.$$

Since ∇ is the Levi-Civita connection on $T\varphi(\mathcal{H})$ induced by $\boldsymbol{g}$, it is torsion free and hence

$$\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\parallel} = \left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}^{\parallel}\right] + \tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{\mathcal{V}} = \left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}^{\parallel}\right] + \tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{\mathcal{V}}^{\parallel} + \tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{\mathcal{V}}^{\perp}.$$

Note that since $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}(\tilde{\varphi}(X,t))$ does not explicitly depend on time, hence, denoting the Lie derivative by $\boldsymbol{L}$, one can write $\left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_{\parallel}\right] = \boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel}$, which is tangent to $\mathcal{S}_t$.[3]

[3]The Lie derivative along the vector field $\boldsymbol{\mathcal{V}}$ is defined as $\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = \frac{d}{dt}\big|_{t=s}\left[\left(\varphi_t \circ \varphi_s^{-1}\right)^* \boldsymbol{\mathcal{V}}_{\parallel}\right]$, where $\varphi_t \circ \varphi_s^{-1}$ is the flow of $\boldsymbol{\mathcal{V}}$.

Following the definition of the second fundamental form, we have $\tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{\mathcal{V}}^{\parallel} = \nabla_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{\mathcal{V}}^{\parallel} + \boldsymbol{\beta}(\boldsymbol{\mathcal{V}}^{\parallel}, \boldsymbol{\mathcal{V}}^{\parallel})\boldsymbol{n}$. We let $\mathcal{V}^n = \tilde{\boldsymbol{g}}(\boldsymbol{\mathcal{V}}, \boldsymbol{n})$, i.e., $\boldsymbol{\mathcal{V}}^{\perp} = \mathcal{V}^n\boldsymbol{n}$. The metric compatibility of $\tilde{\nabla}$ and the fact that $\tilde{\boldsymbol{g}}(\boldsymbol{n}, \boldsymbol{n}) = 1$ imply that for any vector $\boldsymbol{W}$ along $\varphi_t(\mathcal{H})$ in $\mathcal{S}$, we have

$$\frac{d}{dt}\Big(\tilde{\boldsymbol{g}}(\boldsymbol{n}, \boldsymbol{n})\Big) = 2\tilde{\boldsymbol{g}}(\tilde{\nabla}_{\boldsymbol{W}}\boldsymbol{n}, \boldsymbol{n}) = 0\,,$$

i.e., $\tilde{\nabla}_{\boldsymbol{W}}\boldsymbol{n} \in \mathfrak{X}\left(\varphi(\mathcal{H})\right)$. Thus $\tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{n} = -\boldsymbol{g}^{\sharp}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{V}}^{\parallel}$. Therefore

$$\tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{\mathcal{V}}^{\perp} = \tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\left(\mathcal{V}^n\boldsymbol{n}\right) = \tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\left(\mathcal{V}^n\right)\boldsymbol{n} + \mathcal{V}^n\tilde{\nabla}_{\boldsymbol{\mathcal{V}}^{\parallel}}\boldsymbol{n} = \left(d\mathcal{V}^n\cdot\boldsymbol{\mathcal{V}}^{\parallel}\right)\boldsymbol{n} - \mathcal{V}^n\boldsymbol{g}^{\sharp}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{V}}^{\parallel}\,.$$

On the other hand, we have

$$\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp} = \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\left(\mathcal{V}^n\boldsymbol{n}\right) = \frac{d\mathcal{V}^n}{dt}\boldsymbol{n} + \mathcal{V}^n\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{n}\,.$$

However, as observed earlier, $\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{n} \in \mathfrak{X}\left(\varphi(\mathcal{H})\right)$, then, it follows that at $\varphi(X, t)$, one can write

$$\left(\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}\right)^{\perp} = \frac{dV^n}{dt}\boldsymbol{n}\,.$$

Let us now compute $\left(\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}\right)^{\parallel}$. We consider an arbitrary vector field $\boldsymbol{U}$ in $\mathcal{S}$ such that $\boldsymbol{U}$ is tangent to $\mathcal{H}$ in a neighborhood of $\varphi(X, t)$, i.e., $\tilde{\boldsymbol{g}}\left(\boldsymbol{\mathcal{V}}^{\perp}, \boldsymbol{U}\right) = 0$. Hence, $\tilde{\boldsymbol{g}}\left(\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}, \boldsymbol{U}\right) = -\tilde{\boldsymbol{g}}\left(\boldsymbol{\mathcal{V}}^{\perp}, \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\right)$. However, at $\varphi(X, t)$, we have

$$\begin{aligned}\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U} &= [\boldsymbol{\mathcal{V}}, \boldsymbol{U}] + \tilde{\nabla}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}} = [\boldsymbol{\mathcal{V}}, \boldsymbol{U}] + \tilde{\nabla}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}}^{\parallel} + \tilde{\nabla}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}}^{\perp} \\ &= [\boldsymbol{\mathcal{V}}, \boldsymbol{U}] + \nabla_{\boldsymbol{U}}\boldsymbol{V}^{\parallel} + \boldsymbol{\beta}\left(\boldsymbol{V}^{\parallel}, \boldsymbol{U}\right)\boldsymbol{n} + \left(dV^n\cdot\boldsymbol{U}\right)\boldsymbol{n} + V^n\tilde{\nabla}_{\boldsymbol{U}}\boldsymbol{n}\,.\end{aligned}$$

Hence[4]

$$\tilde{\boldsymbol{g}}\left(\boldsymbol{\mathcal{V}}^{\perp}, \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\right) = V^n\boldsymbol{\beta}(\boldsymbol{V}^{\parallel}, \boldsymbol{U}) + V^n\left(dV^n\cdot\boldsymbol{U}\right)\,.$$

Thus, it follows from $\tilde{\boldsymbol{g}}\left(\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}, \boldsymbol{U}\right) = -\tilde{\boldsymbol{g}}\left(\boldsymbol{\mathcal{V}}^{\perp}, \tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\right)$ and by arbitrariness of $\boldsymbol{U}$ that

$$\left(\tilde{\nabla}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}\right)^{\parallel} = -\mathcal{V}^n\boldsymbol{g}^{\sharp}\cdot\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{V}}^{\parallel} - \mathcal{V}^n\left(d\mathcal{V}^n\right)^{\sharp}\,.$$

[4]Note that since the vector $\boldsymbol{U}$ is tangent to $\mathcal{H}$ at $\varphi(X, t)$, the vectors $[\boldsymbol{\mathcal{V}}, \boldsymbol{U}] = \boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}$ and $\tilde{\nabla}_{\boldsymbol{U}}\boldsymbol{n}$ are tangent to $\mathcal{H}$ as well.

Therefore, the parallel and normal components of the material acceleration read

$$\begin{aligned}
\boldsymbol{A}^{\parallel} &= \left[\boldsymbol{V}, \boldsymbol{V}^{\parallel}\right] + \nabla_{\boldsymbol{V}^{\parallel}} \boldsymbol{V}^{\parallel} - 2V^{n} \boldsymbol{g}^{\sharp} \cdot \boldsymbol{\beta} \cdot \boldsymbol{V}^{\parallel} - V^{n} \left(dV^{n}\right)^{\sharp} \,, \\
\boldsymbol{A}^{\perp} &= \left[\frac{dV^{n}}{dt} + \boldsymbol{\beta}\left(\boldsymbol{V}^{\parallel}, \boldsymbol{V}^{\parallel}\right) + d\mathcal{V}^{n} \cdot \boldsymbol{V}^{\parallel}\right] \boldsymbol{n} \,.
\end{aligned}$$

3.3 The governing equations of motion

In this section we derive the governing equations of motion for a shell with an evolving reference configuration: balance of mass, balance of linear and angular momenta, and the kinetic equations of evolution.

Balance of mass We denote the material and spatial surface mass densities (mass per unit area) by ρ and ϱ, respectively, and let $\mathcal{U}$ be any open set in $\mathcal{H}$ with a smooth boundary. We postulate that the motion φ conserves the mass of the system, i.e.

$$\int_{\varphi_t(\mathcal{U})} \varrho ds = \int_{\mathcal{U}} \rho dS. \tag{107}$$

The balance of mass for a motion φ with some external mass input/output can be written as

$$\frac{d}{dt} \int_{\mathcal{U}} \rho dS = \int_{\mathcal{U}} S_m dS, \tag{108}$$

where $S_m = S_m(X, t)$ is a given scalar field characterizing the material rate of change of mass per unit area. Recalling that $dS = \sqrt{\det \boldsymbol{G}}\, dX^1 \wedge dX^2$, it follows from (108) that $\rho = \rho(X, \boldsymbol{G}, t)$, and we find

$$\frac{1}{\sqrt{\det \boldsymbol{G}}} \frac{d}{dt} \left(\sqrt{\det \boldsymbol{G}} \rho\right) = S_m \,, \tag{109}$$

which, using the identity $\frac{d}{dt}\left[\det \boldsymbol{A}(t)\right] = \det \boldsymbol{A}(t) \operatorname{tr}\left[\boldsymbol{A}^{-1}(t) \frac{d}{dt} \boldsymbol{A}(t)\right]$, gives the material local form of the balance of mass for a growing body as

$$\dot{\rho} + \frac{1}{2} \rho \operatorname{tr} \dot{\boldsymbol{G}} = S_m \,, \tag{110}$$

where the dot denotes total time differentiation. Note that if we write the evolving metric as in (97), i.e., $\boldsymbol{G}(X,t) = \boldsymbol{G}^0(X)e^{2\boldsymbol{\omega}(X,t)}$, and use the identity $\det\left(e^{\boldsymbol{A}}\right) = e^{\mathrm{tr}(\boldsymbol{A})}$, then (109) reads

$$\dot{\rho} + \rho\frac{d}{dt}\left[\mathrm{tr}(\boldsymbol{\omega})\right] = S_m\,.$$

Now, since $J = \sqrt{\frac{\det\varphi^*\boldsymbol{g}}{\det\boldsymbol{G}}} = J(X,\varphi,\boldsymbol{G},\boldsymbol{g})$, and $\rho = J\varrho$, it follows from (110) that $\varrho = \varrho(X,\varphi,\boldsymbol{G},\boldsymbol{g},t)$, and we find[5]

$$\dot{\varrho} + \varrho\frac{\dot{J}}{J} + \frac{1}{2}\varrho\,\mathrm{tr}\left(\frac{d\boldsymbol{G}}{dt}\right) = s_m\,, \tag{111}$$

where $s_m(X,t) = \frac{1}{J}S_m(X,t)$ is the spatial rate of change of mass per unit area. Using (101), one can write

$$\dot{J} = \frac{1}{2}\frac{d}{dt}\left(\det\varphi^*\boldsymbol{g}\right)\frac{1}{\sqrt{\det\boldsymbol{G}\det\varphi^*\boldsymbol{g}}} - \frac{1}{2}\frac{d}{dt}\left(\det\boldsymbol{G}\right)\sqrt{\frac{\det\varphi^*\boldsymbol{g}}{\left(\det\boldsymbol{G}\right)^3}}\,.$$

Therefore

$$\frac{\dot{J}}{J} = \frac{1}{2}\mathrm{tr}_{\boldsymbol{C}}\left(\frac{d\varphi^*\boldsymbol{g}}{dt}\right) - \frac{1}{2}\mathrm{tr}\left(\frac{d\boldsymbol{G}}{dt}\right)\,,$$

where $\mathrm{tr}_{\boldsymbol{C}}$ is the trace taken with respect to the metric $\boldsymbol{C}$. Recalling the decomposition $\boldsymbol{v} = \boldsymbol{v}^{\parallel} + v^n\boldsymbol{n}$, we have [50, 86, 37]

$$\varphi_*\frac{d\varphi^*\boldsymbol{g}}{dt} = \boldsymbol{L}_{\boldsymbol{v}}\boldsymbol{g} = \boldsymbol{L}_{\boldsymbol{v}^{\parallel}}\boldsymbol{g} - 2v^n\boldsymbol{\beta}\,,$$

and it follows that

$$\mathrm{tr}_{\boldsymbol{C}}\left(\frac{d\varphi^*\boldsymbol{g}}{dt}\right) = 2\mathrm{div}\boldsymbol{v}^{\parallel} - 2v^n\mathrm{tr}\boldsymbol{\beta}\,,$$

where div denotes the divergence on the surface $\varphi_t(\mathcal{H})$. Therefore, (111) gives the spatial local form of the balance of mass for a growing shell as $\dot{\varrho} + \varrho\mathrm{div}\boldsymbol{v}^{\parallel} - \varrho v^n\mathrm{tr}\boldsymbol{\beta} = s_m$.

[5]Note that (111) can also be obtained from (107) and (108) by writing

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})}\varrho ds = \int_{\mathcal{U}} s_m dS\,.$$

Balance laws Given the right Cauchy-Green deformation tensor $\boldsymbol{C}$ and the extrinsic deformation tensor $\boldsymbol{\Theta}$, the geometry of the deformed surface is uniquely defined (See § 3.2). However, in order to specify the evolution of an element of the deformed surface, we need to know its position φ and its orientation by means of the normal vector field $\boldsymbol{\mathcal{N}}$. Therefore, in the classical theory of nonlinear elasticity of shells, we define the action functional as the map $S : \mathcal{C} \mapsto \mathbb{R}$

$$S(\varphi) = \int_{t_0}^{t_1} \int_{\mathcal{H}} \mathcal{L}(X, \varphi, \boldsymbol{\mathcal{N}}, \dot{\varphi}, \tilde{\boldsymbol{g}} \circ \varphi, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}) dS dt \,,$$

where $\mathcal{L} = \mathcal{L}(X, \varphi, \boldsymbol{\mathcal{N}}, \dot{\varphi}, \tilde{\boldsymbol{g}}, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G})$ is the Lagrangian density per unit surface area.[6] The governing equations of motion follow from Hamilton's principle of least action, which states that the physical motion φ of $\mathcal{H}$ between t_0 and t_1 is a critical point for the action functional, i.e.

$$\delta S(\varphi) = 0 \,.$$

In the present geometric theory of anelastic shells, the material first and second fundamental forms are dynamical variables that vary independently of the motion. Therefore, the action functional is modified to read

$$S(\varphi, \boldsymbol{G}, \boldsymbol{B}) = \int_{t_0}^{t_1} \int_{\mathcal{H}} \mathcal{L}(X, \varphi, \boldsymbol{\mathcal{N}}, \dot{\varphi}, \tilde{\boldsymbol{g}}, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}, \boldsymbol{B}) dS dt \,.$$

We recall that $dS(X,t) = \sqrt{\det \boldsymbol{G}(X,t)}\, dX^1 \wedge dX^2$, and define the Lagrangian density $\mathcal{L}$ by

$$\mathcal{L}(X, \varphi, \boldsymbol{\mathcal{N}}, \dot{\varphi}, \tilde{\boldsymbol{g}}, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}, \boldsymbol{B}) = \frac{1}{2} \rho \tilde{\boldsymbol{g}}(\dot{\varphi}, \dot{\varphi}) - \mathcal{W}(X, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}, \boldsymbol{B}) - \mathcal{V}(X, \varphi, \boldsymbol{\mathcal{N}}, \tilde{\boldsymbol{g}}) \,,$$

where $\mathcal{W} = \mathcal{W}(X, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}, \boldsymbol{B})$ is the elastic energy density per unit surface area (related to the elastic deformation of a surface element), and $\mathcal{V} = \mathcal{V}(X, \varphi, \boldsymbol{\mathcal{N}}, \tilde{\boldsymbol{g}})$ is the potential energy density per unit surface area (related to the position—respectively orientation—of a surface element in the body force—respectively moment—fields).

[6] Since the Lagrangian density is a scalar, it depends on the metrics $\boldsymbol{G}$ and $\tilde{\boldsymbol{g}}$.

Similar to the coordinate chart (X^1, X^2, X^3) previously defined for $\mathcal{B}$, let (x^1, x^2, x^3) be a local coordinate chart for $\mathcal{S}$ such that at any point of the hypersurface $\varphi(\mathcal{H})$, $\{x^1, x^2\}$ is a local coordinate chart for $\varphi(\mathcal{H})$ and the normal vector field $\boldsymbol{n}$ to $\varphi(\mathcal{H})$ is tangent to the coordinate curve x^3. Therefore, the Lagrangian density in this coordinate chart reads

$$\begin{aligned}\mathcal{L}(X, \varphi, \boldsymbol{\mathcal{N}}, \dot{\varphi}, \tilde{\boldsymbol{g}}, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}, \boldsymbol{B}) = \frac{1}{2}\rho g_{ab}\dot{\varphi}^a\dot{\varphi}^b + \frac{1}{2}\rho\left(\dot{\varphi}^n\right)^2 \\ - \mathcal{W}(X, \boldsymbol{C}, \boldsymbol{\Theta}, \boldsymbol{G}, \boldsymbol{B}) - \mathcal{V}(X, \varphi, \boldsymbol{\mathcal{N}}, \tilde{\boldsymbol{g}}) .\end{aligned} \tag{112}$$

Also, because anelasticity is, in general, a non-conservative process, we use Lagrange-d'Alembert's principle, which given non-conservative forces $\boldsymbol{F}_\varphi$, $\boldsymbol{F}_{\boldsymbol{\mathcal{N}}}$, $\boldsymbol{F}_{\boldsymbol{G}}$, and $\boldsymbol{F}_{\boldsymbol{B}}$, states that [49]

$$\delta S(\varphi, \boldsymbol{G}, \boldsymbol{B}) + \int_{t_0}^{t_1}\int_{\mathcal{H}} \left(\boldsymbol{F}_\varphi.\delta\varphi + \boldsymbol{F}_{\boldsymbol{\mathcal{N}}}.\delta\boldsymbol{\mathcal{N}} + \boldsymbol{F}_{\boldsymbol{G}}\!:\!\delta\boldsymbol{G} + \boldsymbol{F}_{\boldsymbol{B}}\!:\!\delta\boldsymbol{B}\right) dSdt = 0 .$$

The sources of these forces depend on the particular underlying source of anelasticity. Here, we assume the existence of a Rayleigh potential $\mathcal{R} = \mathcal{R}(\dot{\varphi}, \dot{\boldsymbol{\mathcal{N}}}, \tilde{\boldsymbol{g}}, \dot{\boldsymbol{G}}, \dot{\boldsymbol{B}}, \boldsymbol{G})$ such that

$$\boldsymbol{F}_\varphi = -\frac{\partial\mathcal{R}}{\partial\dot{\varphi}}, \quad \boldsymbol{F}_{\boldsymbol{\mathcal{N}}} = -\frac{\partial\mathcal{R}}{\partial\dot{\boldsymbol{\mathcal{N}}}}, \quad \boldsymbol{F}_{\boldsymbol{G}} = -\frac{\partial\mathcal{R}}{\partial\dot{\boldsymbol{G}}}, \quad \text{and} \quad \boldsymbol{F}_{\boldsymbol{B}} = -\frac{\partial\mathcal{R}}{\partial\dot{\boldsymbol{B}}} .$$

In the context of our theory, we disregard non-conservative forces due to the variations of position and orientation and assume that it is only due to anelasticity, i.e., we assume that $\mathcal{R} = \mathcal{R}(\dot{\boldsymbol{G}}, \dot{\boldsymbol{B}}, \boldsymbol{G})$.

In order to take variations, we let φ_ϵ be a 1-parameter family of motions such that $\varphi_{0,t} = \varphi_t$.[7] For fixed X and t, we consider the curve $\varphi_{t,X} : \epsilon \mapsto \varphi_{t,X}(\epsilon) := \varphi_{\epsilon,t}(X)$, and define the variation of motion as the spatial vector field given by

$$\delta\varphi(X,t) = T_\epsilon\varphi_{t,X}\left[\partial_\epsilon\right]\Big|_{\epsilon=0} \in T_{\varphi_{\epsilon,t}(X)}\mathcal{S} .$$

Similarly, we let $\boldsymbol{G}_\epsilon$ be a 1-parameter family of material metrics such that $\boldsymbol{G}_{\epsilon=0} = \boldsymbol{G}$ and for fixed X and t, we define the variation of the metric by the material tensor

[7] For fixed X and t, we let $\varphi_{\epsilon,t}(X) := \varphi_\epsilon(X,t)$.

given by

$$\delta \boldsymbol{G}(X,t) = \left.\frac{d\boldsymbol{G}_\epsilon}{d\epsilon}\right|_{\epsilon=0} (X,t)\,.$$

Also, let $\boldsymbol{B}_\epsilon$ be a 1-parameter family of second fundamental forms such that $\boldsymbol{B}_{\epsilon=0} = \boldsymbol{B}$ and for fixed X and t, we define the variation of the second fundamental form by the material tensor given by

$$\delta \boldsymbol{B}(X,t) = \left.\frac{d\boldsymbol{B}_\epsilon}{d\epsilon}\right|_{\epsilon=0} (X,t).$$

It follows from Lagrange-d'Alembert's principle that

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\mathcal{H}} \Bigg(&\frac{\partial \mathcal{L}}{\partial \varphi}.\delta\varphi + \frac{\partial \mathcal{L}}{\partial \boldsymbol{\mathcal{N}}}.\delta\boldsymbol{\mathcal{N}} + \frac{\partial \mathcal{L}}{\partial \dot\varphi}.\delta\dot\varphi + \frac{\partial \mathcal{L}}{\partial \tilde{\boldsymbol{g}}}\!:\!\delta\tilde{\boldsymbol{g}}\circ\varphi + \frac{1}{\sqrt{\det \boldsymbol{G}}}\frac{\partial \sqrt{\det \boldsymbol{G}}\mathcal{L}}{\partial \boldsymbol{G}}\!:\!\delta\boldsymbol{G} \\ &+\frac{\partial \mathcal{L}}{\partial \boldsymbol{C}}\!:\!\delta\boldsymbol{C} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Theta}}\!:\!\delta\boldsymbol{\Theta} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{B}}\!:\!\delta\boldsymbol{B}\Bigg) dSdt = \int_{t_0}^{t_1}\int_{\mathcal{H}} \left(\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}}\!:\!\delta\boldsymbol{G} + \frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{B}}}\!:\!\delta\boldsymbol{B}\right) dSdt\,.\end{aligned} \tag{113}$$

Remark 3.3.1. Note that in taking the variation of the action, the variations of $\boldsymbol{C}$ and $\boldsymbol{\Theta}$ must be such that they satisfy the compatibility equations (102). Since the variation of the action is taken by considering the variation of the deformation mapping φ, the resulting variations of $\boldsymbol{C}$ and $\boldsymbol{\Theta}$ (cf. (116) and (117)) are trivially compatible, i.e., they trivially satisfy the compatibility equations (102). Hence these compatibility equations are not constraints.

If we vary ϵ, for fixed time t and $X \in \mathcal{H}$, the material velocity $\dot\varphi_\epsilon$ and the unit normal $\boldsymbol{\mathcal{N}}\circ\varphi_\epsilon$ lie in $T_{\varphi_{\epsilon,t}(X)}\mathcal{S}$. Therefore, their variations are given by their covariant derivatives along the curve $\varphi_{t,X}$ in $\mathcal{S}$ evaluated at $\epsilon = 0$. By the symmetry lemma (see [17, 56]), we find the variation of the velocity as

$$\begin{aligned}\delta\dot\varphi &= D_{\varphi_{t,X}(\epsilon)}\dot\varphi_\epsilon\big|_{\epsilon=0} = D_{\varphi_{t,X}(\epsilon)}\left[T_t\varphi_{\epsilon,X}\left[\partial_t\right]\big|_{\epsilon=0}\right] \\ &= D_{\varphi_X(t)}\left[T_\epsilon\varphi_{t,X}\left[\partial_\epsilon\right]\right]\big|_{\epsilon=0} = D_{\varphi_X(t)}\delta\varphi =: \frac{D\delta\varphi}{dt}\,.\end{aligned} \tag{114}$$

Following [37], the variation of the unit normal vector field is given by

$$\delta\boldsymbol{\mathcal{N}} = D_{\varphi_{t,X}(\epsilon)}\boldsymbol{\mathcal{N}}_\epsilon\big|_{\epsilon=0} = \tilde\nabla_{\delta\varphi^\parallel}\boldsymbol{\mathcal{N}} - (d(\delta\varphi^n))^\sharp\,,$$

where $\sharp$ denotes the operation of raising indices (sharp operator). In components

$$\delta\mathcal{N}^a = \delta\varphi^b \tilde{\nabla}_{\partial_b}\boldsymbol{\mathcal{N}} - \frac{\partial(\delta\varphi^n)}{\partial x^b} g^{ab}\partial_a = -\left(\delta\varphi^b \beta^a{}_b + \frac{\partial(\delta\varphi^n)}{\partial x^b} g^{ab}\right)\partial_a\,. \tag{115}$$

For any fixed time t and $X \in \mathcal{H}$, the right Cauchy-Green deformation tensor $\boldsymbol{C}_\epsilon$ lies in $S^2T^*_X\mathcal{H}$. Therefore, the variation of $\boldsymbol{C}$ is given by its total derivative with respect to ϵ evaluated at $\epsilon = 0$:

$$\delta\boldsymbol{C} = \left.\frac{d\boldsymbol{C}_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \left.\frac{d}{d\epsilon}\left(\varphi^*_\epsilon \boldsymbol{g}_\epsilon\right)\right|_{\epsilon=0} = \left.\frac{d}{d\epsilon}\left(\varphi^*\varphi_*\varphi^*_\epsilon \boldsymbol{g}_\epsilon\right)\right|_{\epsilon=0} = \varphi^*\boldsymbol{L}_{\delta\varphi}\boldsymbol{g}\,.$$

Note that [50, 86, 37] $\boldsymbol{L}_{\delta\varphi}\boldsymbol{g} = \boldsymbol{L}_{\delta\varphi^\parallel}\boldsymbol{g} - 2\delta\varphi^n\boldsymbol{\beta}$. Hence, we have

$$\delta\boldsymbol{C} = \varphi^*\boldsymbol{L}_{\delta\varphi^\parallel}\boldsymbol{g} - 2\delta\varphi^n\varphi^*\boldsymbol{\beta}\,.$$

In components, it reads

$$\delta C_{AB} = F^a{}_A g_{ac}\delta\varphi^c{}_{|B} + F^b{}_B g_{bc}\delta\varphi^c{}_{|A} - 2\delta\varphi^n F^a{}_A F^b{}_B \beta_{ab}\,. \tag{116}$$

The extrinsic deformation tensor Θ_ϵ lies in the same space $S^2T^*_X\mathcal{H}$ for fixed time t and $X \in \mathcal{H}$. Hence, the variation of Θ is given by its total derivative with respect to ϵ evaluated at $\epsilon = 0$:

$$\delta\boldsymbol{\Theta} = \left.\frac{d\boldsymbol{\Theta}_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \left.\frac{d}{d\epsilon}\left(\varphi^*_\epsilon \boldsymbol{\beta}_\epsilon\right)\right|_{\epsilon=0} = \left.\frac{d}{d\epsilon}\left(\varphi^*\varphi_*\varphi^*_\epsilon \boldsymbol{\beta}_\epsilon\right)\right|_{\epsilon=0} = \varphi^*\boldsymbol{L}_{\delta\varphi}\boldsymbol{\beta}\,.$$

The variation in terms of the Lie derivative of the second fundamental form is given by [86, 37]

$$\boldsymbol{L}_{\delta\varphi}\boldsymbol{\beta} = \boldsymbol{L}_{\delta\varphi^\parallel}\boldsymbol{\beta} - \delta\varphi^n\boldsymbol{\mathcal{C}} + \mathrm{Hess}_{\delta\varphi^n}\,,$$

where $\boldsymbol{\mathcal{C}}$ denotes the third fundamental form of the surface $\varphi_t(\mathcal{H})$ and is defined for $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{X}(\varphi_t(\mathcal{H}))$ by

$$\boldsymbol{\mathcal{C}}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{g}\left(\tilde{\nabla}_{\boldsymbol{x}}\boldsymbol{n}, \tilde{\nabla}_{\boldsymbol{y}}\boldsymbol{n}\right)\,,$$

and Hess_f denotes the Hessian of the scalar-valued function f and is defined for $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{X}(\varphi_t(\mathcal{H}))$ by

$$\mathrm{Hess}_f(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{g}\left(\tilde{\nabla}_{\boldsymbol{x}}(df)^\sharp, \boldsymbol{y}\right)\,.$$

Therefore, we have

$$\delta\boldsymbol{\Theta} = \varphi^*\boldsymbol{L}_{\delta\varphi^{\parallel}}\boldsymbol{\beta} - \delta\varphi^n\varphi^*\boldsymbol{\mathcal{C}} + \varphi^*\mathrm{Hess}_{\delta\varphi^n}\,,$$

or, in components

$$\begin{aligned}\delta\Theta_{AB} &= F^a{}_A F^b{}_B \beta_{ab|c}\delta\varphi^c + F^a{}_A\beta_{ac}\delta\varphi^c{}_{|B} + F^b{}_B\beta_{bc}\delta\varphi^c{}_{|A} \\ &\quad - \delta\varphi^n F^a{}_A F^b{}_B \beta_{ac}\beta_{bd}g^{cd} + F^b{}_A\left(\frac{\partial\delta\varphi^n}{\partial x^b}\right)_{|B}\,.\end{aligned} \tag{117}$$

The variation of the ambient space metric vanishes identically since it is compatible with the connection, i.e.

$$\delta\tilde{\boldsymbol{g}}\circ\varphi = D_{\varphi_{t,X}}\tilde{\boldsymbol{g}}\circ\varphi = \tilde{\nabla}_{\delta\varphi}\tilde{\boldsymbol{g}} = \boldsymbol{0}\,. \tag{118}$$

In order to obtain the balance laws, we fix the first and the second fundamental forms, i.e., $\delta\boldsymbol{G} = \boldsymbol{0}$ and $\delta\boldsymbol{B} = \boldsymbol{0}$ and vary φ. Therefore, following (113) and by arbitrariness of $\delta\varphi$ and $\nabla^{\mathcal{H}}\delta\varphi^n$, we find the following Euler-Lagrange equations[8]

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial\varphi^a} - \left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\mathcal{N}}}\right)_b\beta^b{}_a - \frac{1}{\sqrt{\det\boldsymbol{G}}}\frac{D}{dt}\left[\sqrt{\det\boldsymbol{G}}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^a}\right] - 2\left[\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_A g_{ab}\right]_{|B} \\ &\qquad - \left[\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\beta_{ab}\right]_{|B} - \left[\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right]_{|B}\beta_{ab} = 0\,,\end{aligned} \tag{119a}$$

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial\varphi^n} + \left(\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\mathcal{N}}}\right)_b g^{ab}F^{-A}{}_a\right)_{|A} - \frac{1}{\sqrt{\det\boldsymbol{G}}}\frac{D}{dt}\left[\sqrt{\det\boldsymbol{G}}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^n}\right] - 2\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_A F^a{}_B\beta_{ab} \\ &\qquad - \frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A F^a{}_B\beta_{ac}\beta_{bd}g^{cd} + \left[\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right)_{|B}F^{-D}{}_b\right]_{|D} = 0\,,\end{aligned} \tag{119b}$$

along with the following equations prescribing a vanishing initial velocity vector field

[8] We denote by $F^{-A}{}_a$ the components of $\boldsymbol{F}^{-1}$, the inverse of $\boldsymbol{F}$.

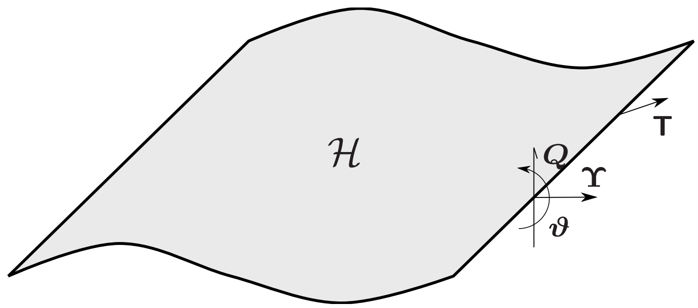


Figure 10: Boundary loads: $\boldsymbol{V}$, $\boldsymbol{\vartheta}$, and $\boldsymbol{Q}$ are the surface traction, the moment, and the shear, respectively. T denotes the outward in-plane normal.

and vanishing boundary conditions for the loading on the boundary $\partial\mathcal{H}$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right|_{t=t_0} = \mathbf{0}\,, \tag{120a}$$

$$\left(2\frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} + 2\frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac}\right) \mathsf{T}_B = 0\,, \tag{120b}$$

$$\left[\left(\frac{\partial \mathcal{L}}{\partial \Theta_{AC}} F^b{}_A\right)_{|C} F^{-B}{}_b + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\mathcal{N}}}\right)_b g^{ab} F^{-B}{}_a\right] \mathsf{T}_B = 0\,, \tag{120c}$$

$$\frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^a{}_A \mathsf{T}_B = 0, \tag{120d}$$

where T is the outward in-plane vector field normal to the boundary $\partial\mathcal{H}$.

Remark 3.3.2. Note that we can modify the Lagrange-d'Alembert's principle in order to prescribe non-vanishing initial and boundary conditions on $\partial\mathcal{H}$. Let $\boldsymbol{V}$ be the boundary surface traction, $\boldsymbol{\vartheta}$ be the boundary moment, $\boldsymbol{Q}$ be the boundary shear, and $\boldsymbol{V}_{t_0}$ be the initial velocity vector field (see Figure 10). We write the Lagrange-d'Alembert's principle as

$$\begin{aligned} &\delta S(\varphi, \boldsymbol{G}, \boldsymbol{B}) + \int_{t_0}^{t_1} \int_{\mathcal{H}} \left(\boldsymbol{F}_{\boldsymbol{G}} \!:\! \delta\boldsymbol{G} + \boldsymbol{F}_{\boldsymbol{B}} \!:\! \delta\boldsymbol{B}\right) dSdt \\ &+ \int_{t_0}^{t_1} \int_{\partial\mathcal{H}} \left(\mathrm{J}V^a g_{ab} \delta\varphi^b + \mathrm{J}\vartheta^a \delta\varphi^n{}_{,A} F^{-A}{}_a + \mathrm{J}Q\delta\varphi^n\right) dLdt + \int_{\mathcal{H}} \rho \boldsymbol{V}_{t_0} . \delta\varphi_{t_0} dS = 0\,, \end{aligned}$$

and from (120) we have

$$
\begin{aligned}
\left.\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right|_{t=t_0} &= \rho \boldsymbol{V}_{t_0}\,,\\
\left(2\frac{\partial \mathcal{L}}{\partial C_{AB}}F^c{}_A g_{ac} + 2\frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^c{}_A \beta_{ac}\right)\mathsf{T}_B &= \mathrm{J} g_{ab} V^b\,,\\
\left[\left(\frac{\partial \mathcal{L}}{\partial \Theta_{AC}}F^b{}_A\right)_{|C} F^{-B}{}_b + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\mathcal{N}}}\right)_b g^{ab}F^{-B}{}_a\right]\mathsf{T}_B &= \mathrm{J}Q\,,\\
\frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^a{}_A \mathsf{T}_B &= \mathrm{J}\vartheta^a\,.
\end{aligned}
$$

We introduce the following surface tensors:

Second Piola-Kirchhoff stress tensor: $\boldsymbol{S} = 2\dfrac{\partial \mathcal{W}}{\partial \boldsymbol{C}}$, in components, $S^{AB} = 2\dfrac{\partial \mathcal{W}}{\partial C_{AB}}$;

First Piola-Kirchhoff stress tensor: $\boldsymbol{P} = 2\boldsymbol{F}\dfrac{\partial \mathcal{W}}{\partial \boldsymbol{C}}$, in components, $P^{bB} = 2\dfrac{\partial \mathcal{W}}{\partial C_{AB}}F^b{}_A$;

Cauchy stress tensor: $\boldsymbol{\sigma} = \dfrac{2}{\mathrm{J}}\boldsymbol{F}\dfrac{\partial \mathcal{W}}{\partial \boldsymbol{C}}\boldsymbol{F}^{\mathsf{T}}$, in components, $\sigma^{ab} = \dfrac{2}{\mathrm{J}}\dfrac{\partial \mathcal{W}}{\partial C_{AB}}F^a{}_A F^b{}_B$;

Material couple stress tensor: $\boldsymbol{M} = \dfrac{\partial \mathcal{W}}{\partial \boldsymbol{\Theta}}$, in components, $M^{AB} = \dfrac{\partial \mathcal{W}}{\partial \Theta_{AB}}$;

Two-point couple stress tensor: $\boldsymbol{\mathcal{M}} = \boldsymbol{F}\dfrac{\partial \mathcal{W}}{\partial \boldsymbol{\Theta}}$, in components, $\mathcal{M}^{bB} = \dfrac{\partial \mathcal{W}}{\partial \Theta_{AB}}F^b{}_A$;

Spatial couple stress tensor: $\boldsymbol{\mu} = \dfrac{1}{\mathrm{J}}\boldsymbol{F}\dfrac{\partial \mathcal{W}}{\partial \boldsymbol{\Theta}}\boldsymbol{F}^{\mathsf{T}}$, in components, $\mu^{ab} = \dfrac{1}{\mathrm{J}}\dfrac{\partial \mathcal{W}}{\partial \Theta_{AB}}F^a{}_A F^b{}_B$.

We further introduce the following notations for the external loads

$$
\begin{aligned}
\text{External body forces}: \boldsymbol{\mathcal{B}} &= -\frac{1}{\rho}\frac{\partial \mathcal{V}}{\partial \varphi}\,,\\
\text{External body moments}: \boldsymbol{\mathcal{L}} &= -\frac{1}{\rho}\frac{\partial \mathcal{V}}{\partial \boldsymbol{\mathcal{N}}}\,.
\end{aligned}
$$

Recalling the balance of mass (109), we have for the Lagrangian density (112)

$$
\frac{1}{\sqrt{\det \boldsymbol{G}}}\frac{D}{dt}\left[\sqrt{\det \boldsymbol{G}}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right] = \rho \tilde{\boldsymbol{g}}\boldsymbol{A} + S_m \tilde{\boldsymbol{g}}\boldsymbol{V}\,.
$$

Therefore, the Euler-Lagrange equations (119) read

$$
\left(P^{aB} + \beta^a{}_b \mathcal{M}^{bB}\right)_{|B} + \beta^a{}_b\left(\mathcal{M}^{bB}{}_{|B}\right) + \rho \mathcal{B}^a - \rho\beta^a{}_b\mathcal{L}^b - S_m\dot{\varphi}^a = \rho A^a\,, \tag{122a}
$$

$$
\begin{aligned}
\left(P^{aB} + \beta^a{}_b \mathcal{M}^{bB}\right)F^c{}_B\beta_{ac} - \left(\mathcal{M}^{bB}{}_{|B}F^{-D}{}_b\right)_{|D} + \rho\mathcal{B}^n&\\
+ \left(\rho\mathcal{L}^a F^{-A}{}_a\right)_{|A} - S_m\dot{\varphi}^n = \rho A^n\,,&
\end{aligned} \tag{122b}
$$

and the vanishing initial and boundary conditions

$$\boldsymbol{V}\big|_{t=t_0} = \mathbf{0}\,, \tag{123a}$$

$$\left(g_{ab}P^{bB} + 2\beta_{ab}\mathcal{M}^{bB}\right)\mathsf{T}_B = 0\,, \tag{123b}$$

$$\left[\mathcal{M}^{bC}{}_{|C}F^{-B}{}_b + \rho\mathcal{L}^a F^{-B}{}_a\right]\mathsf{T}_B = 0\,, \tag{123c}$$

$$\mathcal{M}^{aB}\mathsf{T}_B = 0\,. \tag{123d}$$

We apply the Piola transform[9] to (122) and (123) and obtain the spatial version of the balance of linear momenta as

$$\left(\sigma^{ac} + \beta^a{}_b\mu^{bc}\right)_{|c} + \beta^a{}_b\mu^{bc}{}_{|c} + \varrho\mathcal{B}^a - \varrho\beta^a{}_b\mathcal{L}^b - s_m\dot{\varphi}^a = \varrho A^a\,, \tag{124a}$$

$$\left(\sigma^{ac} + \beta^a{}_b\mu^{bc}\right)\beta_{ac} - \mu^{ab}{}_{|ab} + \varrho\mathcal{B}^n + \left(\varrho\mathcal{L}^a\right)_{|a} - s_m\dot{\varphi}^n = \varrho A^n\,, \tag{124b}$$

and the vanishing initial and boundary conditions

$$\boldsymbol{V}\big|_{t=t_0} = \mathbf{0}\,, \tag{125a}$$

$$\left(\sigma^{ac} + 2\beta^a{}_b\mu^{bc}\right)\mathsf{t}_c = 0\,, \tag{125b}$$

$$\left(\mu^{ab}{}_{|b} + \varrho\mathcal{L}^a\right)\mathsf{t}_a = 0\,, \tag{125c}$$

$$\mu^{ab}\mathsf{t}_b = 0\,, \tag{125d}$$

where $\mathbf{t}$ is the outward in-plane vector field normal to the boundary $\partial\varphi_t(\mathcal{H})$. By pulling back the system of equations (124) with the mapping φ, we obtain the Euler-Lagrange equations in the convected manifold $(\mathcal{H},\boldsymbol{C})$[10] in terms of the convected stress tensor $\boldsymbol{\Sigma} = \varphi^*\boldsymbol{\sigma} = \boldsymbol{S}/\mathrm{J}$ and the convected couple stress tensor $\boldsymbol{\Lambda} = \varphi^*\boldsymbol{\mu} = \boldsymbol{M}/\mathrm{J}$. If we denote by a double stroke $||$ the covariant derivative corresponding to

[9] Recall the Piola identity $\left(JF^{-A}{}_a\right)_{|A} = 0$.

[10] We define the convected manifold to be the material manifold $\mathcal{H}$ equipped with the right Cauchy-Green deformation tensor $\boldsymbol{C}$.

the Levi-Civita connection of $(\mathcal{H}, \boldsymbol{C})$, the convected Euler-Lagrange equation read[11]

$$\begin{aligned}\left(\Sigma^{AB} + C^{-AC}\Theta_{CD}\Lambda^{DB}\right)_{\|B} + C^{-AC}\Theta_{CD}\Lambda^{DB}{}_{\|B} + \varrho F^{-A}{}_a\mathcal{B}^a \\ -\varrho C^{-AC}\Theta_{CD}F^{-D}{}_a\mathcal{L}^a - s_m F^{-A}{}_a\dot{\varphi}^a = \varrho F^{-A}{}_a A^a\,,\end{aligned} \tag{126a}$$

$$\left(\Sigma^{AB} + C^{-AC}\Theta_{CD}\Lambda^{DB}\right)\Theta_{AB} - \Lambda^{AB}{}_{\|AB} + \varrho\mathcal{B}^n + \left(\varrho\mathcal{L}^a F^{-A}{}_a\right)_{\|A} - s_m\dot{\varphi}^n = \varrho A^n\,, \tag{126b}$$

and the vanishing initial and boundary conditions

$$\boldsymbol{V}|_{t=t_0} = \boldsymbol{0}\,, \tag{127a}$$

$$\left(\Sigma^{AB} + 2C^{-AC}\Theta_{CD}\Lambda^{DB}\right)\mathsf{T}_B = 0\,, \tag{127b}$$

$$\left(\Lambda^{AB}{}_{\|B} + \varrho F^{-A}{}_a\mathcal{L}^a\right)\mathsf{T}_A = 0\,, \tag{127c}$$

$$\Lambda^{AB}\mathsf{T}_B = 0\,. \tag{127d}$$

Recall that in terms of the deformation mapping $\varphi : \mathcal{H} \to \mathbb{R}^3$, we can write the components of $\boldsymbol{C}$ and $\boldsymbol{\Theta}$ in a local chart $\{X, Y\}$ of $\mathcal{H}$ as follow

$$\begin{aligned}C_{AB} &= \varphi_{,A}\cdot\varphi_{,B}\,,\\ \Theta_{AB} &= \varphi_{,AB}\cdot\frac{\varphi_{,X}\times\varphi_{,Y}}{\|\varphi_{,X}\times\varphi_{,Y}\|}\,.\end{aligned}$$

where $\cdot$, $\times$, and $\|.\|$, respectively, denote the dot product, the cross product, and the standard norm in $\mathbb{R}^3$.

Given a constitutive relation, the stress and the couple stress tensors can be written in terms of the first and the second fundamental forms of the deformed surface. On the other hand, the first and the second fundamental forms of the deformed surface can be written in terms of the motion φ such that the compatibility equations (102) are trivially satisfied. Therefore, the system of equations (122) (or (124), or (126)) is a set of three equations for three unknowns (the three components of the motion), and together with the initial and boundary conditions (123) (or (125), or (127)), they form the complete set of governing equations for the morphoelastic shell problem.

[11]The components of $\boldsymbol{C}^{-1}$, the inverse of $\boldsymbol{C}$, are denoted by C^{-AB}.

Remark 3.3.3. Note that both systems of equations (122) and (124) reduce to the elastic shell equilibrium equations for a zero-acceleration motion in the absence of any external mass exchange ($s_m = 0$) and dissipation ($\mathcal{R} = 0$). See [12] (Equations (2.8)), [74] (Equations (55) and (56)), [53] (Equations (5.36)), and [39] (Equations (6.3) and (6.4)) where the shell problem is described by a system of three equations involving six stress and couple stress components. Note that an alternative description is provided by a system of six equations involving ten stress and couple stress components, see [30] (Equations (3.6) and (3.11)) and [21] (Equations (26.6), (26.7) and (26.10)).

Remark 3.3.4. Following the definitions of the surface tensors and based on the symmetry of the right Cauchy-Green tensor and the extrinsic deformation tensor, we have the following symmetries for the stress tensors, which are the local forms of the balance of angular momenta

$$\begin{aligned}
&\boldsymbol{S}^\mathsf{T} = \boldsymbol{S}\,, \quad \boldsymbol{\Sigma}^\mathsf{T} = \boldsymbol{\Sigma}\,, \quad \boldsymbol{P}\boldsymbol{F}^\mathsf{T} = \boldsymbol{P}^\mathsf{T}\boldsymbol{F}\,, \quad \boldsymbol{\sigma}^\mathsf{T} = \boldsymbol{\sigma}\,,\\
&\boldsymbol{M}^\mathsf{T} = \boldsymbol{M}\,, \quad \boldsymbol{\Lambda}^\mathsf{T} = \boldsymbol{\Lambda}\,, \quad \boldsymbol{\mathcal{M}}\boldsymbol{F}^\mathsf{T} = \boldsymbol{\mathcal{M}}^\mathsf{T}\boldsymbol{F}\,, \quad \boldsymbol{\mu}^\mathsf{T} = \boldsymbol{\mu}\,.
\end{aligned} \tag{128}$$

Kinetic equations of evolution If we fix the motion, i.e., $\delta\varphi = 0$, and vary the first and the second fundamental forms, we obtain by arbitrariness of $\delta\boldsymbol{G}$ and $\delta\boldsymbol{B}$ from (113) the following kinetic equations for the evolution of the first and the second fundamental forms of $\mathcal{H}$:

$$\begin{aligned}
\frac{1}{\sqrt{\det \boldsymbol{G}}}\frac{\partial(\sqrt{\det \boldsymbol{G}}\;\mathcal{L})}{\partial \boldsymbol{G}} &= \frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}}\,,\\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{B}} &= \frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{B}}}\,.
\end{aligned}$$

Therefore, we find the following for the Lagrangian density (112)

$$\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}} = \frac{1}{2}\left(\frac{1}{2}\rho\tilde{\boldsymbol{g}}(\dot{\varphi},\dot{\varphi}) - \mathcal{W} - \mathcal{V}\right)\boldsymbol{G}^\sharp - \frac{\partial \mathcal{W}}{\partial \boldsymbol{G}}\,, \tag{130a}$$

$$\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{B}}} = -\frac{\partial \mathcal{W}}{\partial \boldsymbol{B}}\,. \tag{130b}$$

Assuming the existence of a Rayleigh potential $\mathcal{R} = \mathcal{R}(\dot{\boldsymbol{G}}, \dot{\boldsymbol{B}}, \boldsymbol{G})$, we introduce a variational characterization for the variation of energy in the shell due to growth. The system of equations (130) provide a coupling of the rate of change of the first and the second fundamental forms of $\mathcal{H}$ with its current state of deformation through the elastic energy density $\mathcal{W}$ of the material. Therefore, the evolution of the geometry of the shell, i.e., the growth of the morphoelastic shell, is governed by (130).

Remark 3.3.5. [89] discussed the kinetic equation for the evolving metric in the case of bulk growth for three-dimensional nonlinear elasticity. Note, however, that there was a missing term in equation (2.179), which should be corrected to read

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} dV dt = \int_{t_0}^{t_1} \int_{\mathcal{B}} \left[\delta \mathcal{L} + \frac{1}{2} \mathcal{L} \mathrm{tr}(\delta \boldsymbol{G}) \right] dV dt \, .$$

The kinetic equation (2.181) in [89] should also be corrected to read

$$\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{G}} + \frac{1}{2} \mathcal{L} \boldsymbol{G}^{\sharp} \, . \tag{131}$$

Ignoring inertial forces and in the absence of body forces, (131) reads

$$\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}} = -\frac{\partial \mathcal{W}}{\partial \boldsymbol{G}} - \frac{1}{2} \mathcal{W} \boldsymbol{G}^{\sharp} \, .$$

Following the material covariance of nonlinear elasticity, [47, 96] proved the following relation:

$$\frac{1}{2} \boldsymbol{S}_o \boldsymbol{G} = \frac{\partial \mathcal{W}}{\partial \boldsymbol{G}} \boldsymbol{G} + \frac{\partial \mathcal{W}}{\partial \boldsymbol{C}} \boldsymbol{C} \, ,$$

where $\boldsymbol{S}_o$ is a stress-like tensor conjugate to $\dot{\boldsymbol{G}}$. $\boldsymbol{S}_o$ is associated with the material evolution and is a measure of anisotropy of the medium: $\boldsymbol{S}_o = \boldsymbol{0}$ for an isotropic material. Therefore, (131) can be rewritten as

$$\frac{\partial \mathcal{R}}{\partial \dot{\boldsymbol{G}}} \boldsymbol{G} = -\frac{1}{2} \mathcal{W} \boldsymbol{G}^{\sharp} \boldsymbol{G} + \frac{1}{2} \boldsymbol{S} \boldsymbol{C} - \frac{1}{2} \boldsymbol{S}_o \boldsymbol{G} \, . \tag{132}$$

In the context of the multiplicative decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_g$, the kinetic equation coupling the evolution of growth and stress is written in terms of the growth tensor $\boldsymbol{F}_g$. [26] derived it using the so-called principle of maximum entropy production rate, and [2, 61] used the Clausius-Duhem inequality. Note that these equations are both equivalent and similar in form to (132).

Example 3.3.1. As an example, we consider the following Rayleigh potential

$$\mathcal{R}(\dot{\boldsymbol{G}}, \dot{\boldsymbol{B}}, \boldsymbol{G}) = \alpha_1 \mathrm{tr}(\dot{\boldsymbol{G}}) + \alpha_2 \mathrm{tr}(\dot{\boldsymbol{G}}^2) + \beta_1 \mathrm{tr}(\dot{\boldsymbol{B}}) + \beta_2 \mathrm{tr}(\dot{\boldsymbol{B}}^2) \,. \tag{133}$$

In components

$$\mathcal{R}(\dot{\boldsymbol{G}}, \dot{\boldsymbol{B}}, \boldsymbol{G}) = \alpha_1 \dot{G}_{AB} G^{AB} + \alpha_2 \dot{G}_{AB} \dot{G}_{CD} G^{CA} G^{DB} + \beta_1 \dot{B}_{AB} G^{AB} + \beta_2 \dot{B}_{AB} \dot{B}_{CD} G^{CA} G^{DB}.$$

Therefore, if we assume a static shell in the absence of body forces and moments, the kinetic equations (130) read

$$\dot{\boldsymbol{G}} = -\frac{\alpha_1}{2\alpha_2}\boldsymbol{G} - \frac{1}{4\alpha_2}\mathcal{W}\boldsymbol{G} - \frac{1}{2\alpha_2}\boldsymbol{G}\frac{\partial \mathcal{W}}{\partial \boldsymbol{G}}\boldsymbol{G} \,, \tag{134a}$$

$$\dot{\boldsymbol{B}} = -\frac{\beta_1}{2\beta_2}\boldsymbol{G} - \frac{1}{2\beta_2}\boldsymbol{G}\frac{\partial \mathcal{W}}{\partial \boldsymbol{B}}\boldsymbol{G} \,. \tag{134b}$$

In components

$$\begin{aligned}
\dot{G}_{AB} &= -\frac{\alpha_1}{2\alpha_2} G_{AB} - \frac{1}{4\alpha_2}\mathcal{W} G_{AB} - \frac{1}{2\alpha_2} G_{AC} \frac{\partial \mathcal{W}}{\partial G_{DC}} G_{DB} \,, \\
\dot{B}_{AB} &= -\frac{\beta_1}{2\beta_2} G_{AB} - \frac{1}{2\beta_2} G_{AC} \frac{\partial \mathcal{W}}{\partial B_{DC}} G_{DB} \,.
\end{aligned}$$

We assume a Saint Venant-Kirchhoff constitutive model, for which the strain energy density $\mathcal{W}$ is given by[12]

$$\begin{aligned}
\mathcal{W} &= \frac{h}{4}\left\{\mu \mathrm{tr}\left[(\boldsymbol{C}-\boldsymbol{G})^2\right] + \frac{\mu\lambda}{2\mu+\lambda}\left[\mathrm{tr}\,(\boldsymbol{C}-\boldsymbol{G})\right]^2\right\} \\
&\quad + \frac{h^3}{12}\left\{\mu \mathrm{tr}\left[(\boldsymbol{\Theta}-\boldsymbol{B})^2\right] + \frac{\mu\lambda}{2\mu+\lambda}\left[\mathrm{tr}\,(\boldsymbol{\Theta}-\boldsymbol{B})\right]^2\right\} \\
&= \frac{Eh}{8(1+\nu)}\left\{\mathrm{tr}\left[(\boldsymbol{C}-\boldsymbol{G})^2\right] + \frac{\nu}{1-\nu}\left[\mathrm{tr}\,(\boldsymbol{C}-\boldsymbol{G})\right]^2\right\} \\
&\quad + \frac{Eh^3}{24(1+\nu)}\left\{\mathrm{tr}\left[(\boldsymbol{\Theta}-\boldsymbol{B})^2\right] + \frac{\nu}{1-\nu}\left[\mathrm{tr}\,(\boldsymbol{\Theta}-\boldsymbol{B})\right]^2\right\} \,,
\end{aligned} \tag{136}$$

[12] For details on the derivation of the Saint Venant-Kirchhoff shell model, see [22, 43, 45, 51, 46, 24, 25, 23].

where λ and μ are respectively Lamé's first and second parameters, E is the Young's modulus, and ν is Poisson's ratio. Therefore, for a Saint Venant-Kirchhoff constitutive model, the kinetic equations (134) read

$$\begin{aligned}
\dot{G}_{AB} =& -\frac{\alpha_1}{2\alpha_2}G_{AB} - \frac{h}{16\alpha_2}\left[\mu\left(C_{KL}-G_{KL}\right)\left(C^{KL}-G^{KL}\right) + \frac{\mu\lambda}{2\mu+\lambda}\left({C^K}_K - 2\right)^2\right]G_{AB} \\
&- \frac{h^3}{48\alpha_2}\left[\mu\left(\Theta_{KL}-B_{KL}\right)\left(\Theta^{KL}-B^{KL}\right) + \frac{\mu\lambda}{2\mu+\lambda}\left({\Theta^K}_K - {B^K}_K\right)^2\right]G_{AB} \\
&+ \frac{h}{4\alpha_2}\left[\mu C_{AK}\left(C_{LB}-G_{LB}\right)G^{KL} + \frac{\mu\lambda}{2\mu+\lambda}\left({C^K}_K - 2\right)C_{AB}\right] \\
&+ \frac{h^3}{12\alpha_2}\left[\mu\left(\Theta_{AK}-B_{AK}\right)\left(\Theta_{LB}-B_{LB}\right)G^{KL} + \frac{\mu\lambda}{2\mu+\lambda}\left({\Theta^K}_K - {B^K}_K\right)\left(\Theta_{AB}-B_{AB}\right)\right], \\
\dot{B}_{AB} =& -\frac{\beta_1}{2\beta_2}G_{AB} + \frac{h^3}{12\beta_2}\left[\mu\left(\Theta_{AB}-B_{AB}\right) + \frac{\mu\lambda}{2\mu+\lambda}\left({\Theta^K}_K - {B^K}_K\right)G_{AB}\right].
\end{aligned}$$

3.4 Example: Morphoelastic Shells

As an application of the proposed geometric theory of anelastic shells, we study examples in the case when anelasticity is due to bulk growth and remodelling of the shell. See our work on morphoelastic shells [72] for more details. We look at the stress-free growth of an initially planar sheet, and study the residual stress and geometry evolution of a morphoelastic infinitely long circular cylindrical shell subject to an internal pressure and a morphoelastic initially planar circular disk.

3.4.1 Stress-free growth fields for an initially flat simply-connected shell

We consider an initially flat thin shell $\mathcal{B}$ such that its mid-surface $\mathcal{H}$ is simply-connected. Let (X,Y,Z) be the standard coordinate chart for $\mathbb{R}^3$ such that the hyperplane $X^3=0$ contains $\mathcal{H}$. We assume that the morphoelastic shell is undergoing a growth field that is modeled by the following evolving metric for $\mathcal{B}$:

$$\bar{\boldsymbol{G}} = \begin{pmatrix} e^{2\bar{\omega}_X(X,Y,Z,t)} & 0 & 0 \\ 0 & e^{2\bar{\omega}_Y(X,Y,Z,t)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which corresponds to the following evolving first and second fundamental forms for the mid-surface $\mathcal{H}$:

$$\boldsymbol{G} = \begin{pmatrix} e^{2\omega_X(X,Y,t)} & 0 \\ 0 & e^{2\omega_Y(X,Y,t)} \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} -K_X(X,Y,t)e^{2\omega_X(X,Y,t)} & 0 \\ 0 & -K_Y(X,Y,t)e^{2\omega_Y(X,Y,t)} \end{pmatrix},$$

where $\omega_A(X,Y,t) = \bar{\omega}_A(X,Y,0,t)$ and $K_A(X,Y,t) = \frac{\partial \bar{\omega}_A}{\partial Z}(X,Y,0,t)$ for $A = X,Y$. Following §3.2, the growth of a simply-connected shell is stress-free if and only if the relations (104) hold. In the case of an initially flat morphoelastic simply-connected shell, the growth is stress-free if and only if

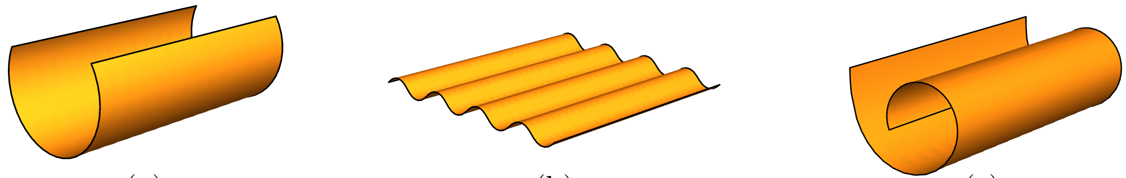
$$e^{2\omega_X}\left[\left(\frac{\partial \omega_Y}{\partial Y} - \frac{\partial \omega_X}{\partial Y}\right)\frac{\partial \omega_X}{\partial Y} - \frac{\partial^2 \omega_X}{\partial Y^2}\right] + e^{2\omega_Y}\left[\left(\frac{\partial \omega_X}{\partial X} - \frac{\partial \omega_Y}{\partial X}\right)\frac{\partial \omega_Y}{\partial X} - \frac{\partial^2 \omega_Y}{\partial X^2}\right]$$
$$= K_X K_Y e^{2\omega_X} e^{2\omega_Y},$$
$$\frac{\partial K_X}{\partial Y} = (K_Y - K_X)\frac{\partial \omega_X}{\partial Y},$$
$$\frac{\partial K_Y}{\partial X} = (K_X - K_Y)\frac{\partial \omega_Y}{\partial X}.$$

Now we consider the following simplifying assumptions:

- If we assume that the in-plane growth is uniform, i.e., $\omega_A = \omega_A(t)$ for $A = X,Y$, we find that the growth is stress-free if and only if $K_X = K_X(X,t)$, $K_Y = K_Y(Y,t)$, and $K_X K_Y = 0$. This case includes the stress-free growth of a planar sheet into a cylindrical portion. See Figure 11 for examples of evolutions of planar sheets into flat surfaces with stress-free growth.

- If we assume that the evolving curvatures K_X and K_Y are uniform, i.e., $K_A = K_A(t)$ for $A = X,Y$, we distinguish the following cases:

  - If $K_X \neq K_Y$, then the growth is stress-free if and only if ω_X and ω_Y are uniform and $K_X K_Y = 0$. This is precisely the case of a planar sheet evolving to a cylindrical portion with a stress-free growth (see Figure 11-a).



(a) (b) (c)

Figure 11: Visualization of a few stress-free material evolutions of an initially planar sheet with the prescribed evolving fundamental forms such that the in-plane growth is uniform, i.e., $\omega_A = \omega_A(t)$ for $A = X, Y$, the Gaussian curvature is vanishing, i.e., $K_X K_Y = 0$, and the non-zero principal curvature is such that $K_X = K_X(X,t)$ or $K_Y = K_Y(Y,t)$. We assume for these figures that $K_Y = 0$ and (a) $K_X = K_X(t)$ to grow to a cylindrical portion, (b) $K_X(X,t) = k_1(t)\sin(k_2(t)X)$, where $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are some arbitrary functions of time resulting in a sheet with sinusoidal rippling, and (c) $K_X(X,t) = k(t)\sqrt{X}$, for $X > 0$, where $k = k(t)$ is some arbitrary function of time.

– If $K = K_X = K_Y$, then the growth is stress-free if and only if

$$e^{2\omega_X}\left[\left(\frac{\partial \omega_Y}{\partial Y} - \frac{\partial \omega_X}{\partial Y}\right)\frac{\partial \omega_X}{\partial Y} - \frac{\partial^2 \omega_X}{\partial Y^2}\right] + e^{2\omega_Y}\left[\left(\frac{\partial \omega_X}{\partial X} - \frac{\partial \omega_Y}{\partial X}\right)\frac{\partial \omega_Y}{\partial X} - \frac{\partial^2 \omega_Y}{\partial X^2}\right] = K^2 e^{2\omega_X} e^{2\omega_Y}.$$

- If we assume that the in-plane growth is isotropic, i.e., $\omega = \omega_X = \omega_Y$, we distinguish the following cases:

  – If $K = K_X = K_Y$, then the growth is stress-free if and only if K is uniform and $\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} = -K^2 e^{2\omega}$. In particular, if $K_X = K_Y = 0$, then the growth is stress-free if and only if ω is harmonic. See Example 3.4.1 and Figure 12 for examples of such a stress-free growth.

– If $K_X \neq K_Y$, then the growth is stress-free if and only if

$$\begin{aligned}
\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} &= -K_X K_Y e^{2\omega} \,, \\
\frac{\partial K_X}{\partial Y} &= (K_Y - K_X) \frac{\partial \omega}{\partial Y} \,, \\
\frac{\partial K_Y}{\partial X} &= (K_X - K_Y) \frac{\partial \omega}{\partial X} \,.
\end{aligned}$$

See Example 3.4.2 and Figure 13 for examples of such a stress-free growth assuming that $K_X = -K_Y$.

Example 3.4.1. In this example, we consider a morphoelastic initially planar square sheet in the XY-plane such that center of the shell coincides with the origin of the coordinate system and the sides of the shell are parallel to the X and Y axes. We assume that both the in-plane and the out-of-plane growths are isotropic, i.e., $\omega = \omega_X = \omega_Y$, and $K = K_X = K_Y$. Therefore, the growth is stress-free if and only if $K = K(t)$ is a uniform arbitrary function of time and ω is such that

$$\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} = -K^2 e^{2\omega} \,. \tag{140}$$

Following [67], a solution of (140) is given by

$$\omega(X, Y, t) = \frac{1}{2} \ln \left(\frac{A^2(t) + B^2(t)}{K^2(t) \cosh^2 \left[C(t) + A(t) X + B(t) Y \right]} \right) \,,$$

for some arbitrary functions of time $A = A(t)$, $B = B(t)$, $C = C(t)$, and $K = K(t)$. Therefore, the first and the second fundamental forms read

$$\boldsymbol{G} = \frac{A^2(t) + B^2(t)}{K^2(t) \cosh^2 \left[C(t) + A(t) X + B(t) Y \right]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,,$$

$$\boldsymbol{B} = -\frac{A^2(t) + B^2(t)}{K(t) \cosh^2 \left[C(t) + A(t) X + B(t) Y \right]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,.$$

It is readily seen that every point of the surface is an umbilical point (the principal curvatures are equal to $K(t)$). Therefore, at a given time t , we have a surface of

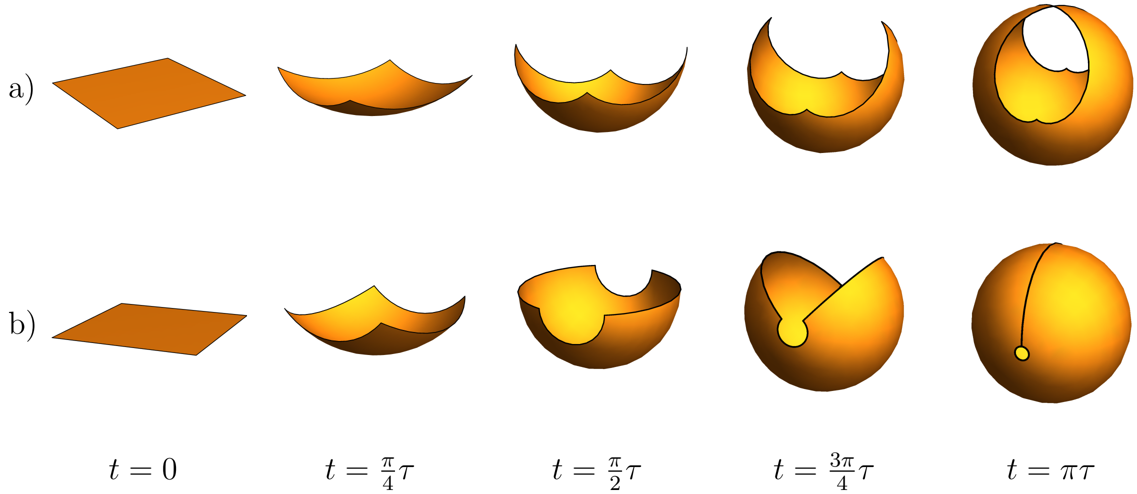


Figure 12: Example 3.4.1: Visualization of the stress-free material evolution of an initially planar sheet with the prescribed evolving fundamental forms (142) and (143), shown, respectively, in a) and b), at different times. Note that the change of shape of the shell is due to growth and not stretch; such an evolution is stress-free.

constant non-negative curvature $K^2(t)$, and hence it is either a planar ($K = 0$) or a spherical ($K > 0$) surface of radius $1/K(t)$ (see Figure 12).

The functions $A = A(t)$, $B = B(t)$, $C = C(t)$, and $K = K(t)$ define the time evolution of the first and the second fundamental forms. Given a constitutive equation for the material, their evolution can subsequently be obtained from the kinetic equations (130) governing the evolution of growth. As an example, and for the purpose of illustrating the non-trivial evolution of the initially planar shell as a result of a stress-free growth, we consider the following cases:

- We assume that $A(t) = t/\tau$, $B(t) = t/\tau$, $C(t) = 0$ and $K_0(t) = \sqrt{2}t/\tau$, where τ is some growth characteristic time. Hence, $\omega(X, Y, t) = -\ln\{\cosh[(X + Y)t/\tau]\}$, and at $t = 0$: $\omega(X, Y, 0) = 0$ and $K(0) = 0$. Therefore, we have the following evolving first and second fundamental forms:

$$\boldsymbol{G} = \frac{1}{\cosh^2\left[(X+Y)t/\tau\right]}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{B} = -\frac{\sqrt{2}t/\tau}{\cosh^2\left[(X+Y)t/\tau\right]}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{142}$$

- We assume that $A(t) = 2t/\tau$, $B(t) = 0$, $C(t) = 0$ and $K_0(t) = 2t/\tau$. It follows that $\omega(X,Y,t) = -\ln\{\cosh\left[2Xt/\tau\right]\}$, such that at $t=0$ they satisfy $\omega(X,Y,0) = 0$ and $K(X,Y,0) = 0$. Therefore, we have the following evolving first and second fundamental forms:

$$\boldsymbol{G} = \frac{1}{\cosh^2\left[2Xt/\tau\right]}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{B} = -\frac{2t/\tau}{\cosh^2\left[2Xt/\tau\right]}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{143}$$

We visualize in Figure 12, the evolution of the initially planar sheet with the prescribed fundamental forms (142) and (143).

Example 3.4.2. In this example, we consider a morphoelastic initially flat square sheet in the XY-plane such that the center of the shell coincides with the origin of the coordinate system and the sides of the shell are parallel to the X and Y axes. We assume that the in-plane growth is isotropic, i.e., $\omega = \omega_X = \omega_Y$, and assume that $K = K_X = -K_Y \neq 0$. We look for ω and K such that the growth is stress-free, i.e., such that

$$\frac{\partial^2\omega}{\partial Y^2} + \frac{\partial^2\omega}{\partial X^2} = K^2 e^{2\omega}, \tag{144a}$$

$$\frac{\partial K}{\partial Y} = -2K\frac{\partial\omega}{\partial Y}, \tag{144b}$$

$$\frac{\partial K}{\partial X} = -2K\frac{\partial\omega}{\partial X}. \tag{144c}$$

It follows from (144b) and (144c) that $K(X,Y,t) = K_o(t)e^{-2\omega(X,Y,t)}$ for some arbitrary function of time $K_o = K_o(t)$. Therefore, (144a) now reads

$$\frac{\partial^2\omega}{\partial Y^2} + \frac{\partial^2\omega}{\partial X^2} = K_o^2 e^{-2\omega}. \tag{145}$$

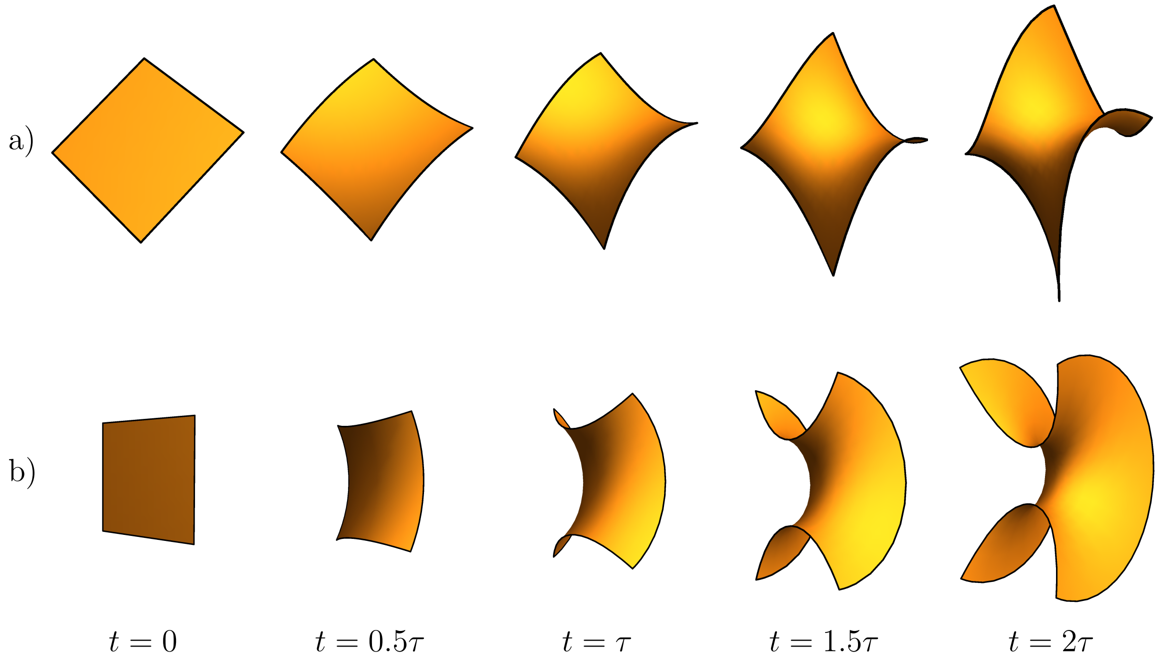


Figure 13: Example 3.4.2: Visualization of the stress-free material evolution of an initially planar sheet with the prescribed evolving fundamental forms (146) and (147), shown, respectively, in a) and b), at different times. Note that the change of shape of the shell is due to growth and not stretch; such an evolution is stress-free.

Following [67], a solution for (145) is given by

$$\omega(X,Y,t) = -\frac{1}{2}\ln\left(\frac{A^2(t)+B^2(t)}{K_o^2(t)\cosh^2\left(C(t)+A(t)X+B(t)Y\right)}\right)\,,$$

for some arbitrary functions of time $A = A(t)$, $B = B(t)$, $C = C(t)$, and $K_o(t)$. As an example, and for the purpose of illustrating the non-trivial form the initially flat shell could adopt as a result of a stress-free growth, we consider the following cases:

- We assume that $A(t) = t/\tau$, $B(t) = t/\tau$, $C(t) = 0$ and $K_0(t) = \sqrt{2}t/\tau$. It follows that

  $$\omega(X,Y,t) = \ln\left\{\cosh\left[(X+Y)t/\tau\right]\right\}\,,\quad K(X,Y,t) = \frac{\sqrt{2}t/\tau}{\cosh^2\left[(X+Y)t/\tau\right]}\,,$$

  such that at $t=0$ they satisfy $\omega(X,Y,0)=0$ and $K(X,Y,0)=0$. Therefore, we have the following evolving first and second fundamental forms:

$$\boldsymbol{G} = \cosh^2\left[(X+Y)t/\tau\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \sqrt{2}t/\tau \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{146}$$

- We assume that $A(t) = 2t/\tau$, $B(t) = 0$, $C(t) = 0$ and $K_0(t) = 2t/\tau$. It follows that

  $$\omega(X,Y,t) = \ln\left\{\cosh\left[2Xt/\tau\right]\right\}, \quad K(X,Y,t) = \frac{2t/\tau}{\cosh^2\left[2Xt/\tau\right]},$$

  such that at $t=0$ they satisfy $\omega(X,Y,0) = 0$ and $K(X,Y,0) = 0$. Therefore, we have the following evolving first and second fundamental forms:

  $$\boldsymbol{G} = \cosh^2\left[2Xt/\tau\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{B} = 2t/\tau \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{147}$$

We visualize in Figure 13, the evolution of the initially planar sheet with the prescribed fundamental forms (146) and (147).

Remark 3.4.1. In the previous examples we obtained the first and the second fundamental forms for stress-free growth fields. Recall that a growth field leaves the surface stress-free if and only if it is embeddable in $\mathbb{R}^3$. Therefore, given a surface $(\mathcal{H}, \boldsymbol{G}, \boldsymbol{B})$ with a stress-free growth field, we can find an isometric embedding of it in $\mathbb{R}^3$ by integrating for the $\mathbb{R}^3$-valued function $\boldsymbol{f}$, the following system of partial differential equations written in a local chart $\{X, Y\}$ of $\mathcal{H}$:

$$\boldsymbol{f}_{,AB} = \Gamma^C{}_{AB}\boldsymbol{f}_{,C} + \boldsymbol{B}_{AB}\boldsymbol{N}, \tag{148}$$

where $\boldsymbol{N} = \frac{\boldsymbol{f}_{,X}\times\boldsymbol{f}_{,Y}}{\|\boldsymbol{f}_{,X}\times\boldsymbol{f}_{,Y}\|}$, $\times$ and $\|.\|$, respectively, denote the cross product and the standard norm in $\mathbb{R}^3$, $\Gamma^C{}_{AB}$ is the Christoffel symbol of the Levi-Civita connection $\nabla^{\mathcal{H}}$ in the local chart $\{X^1, X^2\}$. The integrability conditions for (148) is the equality of the mixed partial for $\boldsymbol{f}$, i.e., $\boldsymbol{f}_{,XY} = \boldsymbol{f}_{,YX}$, which is equivalent to the stress-free growth compatibility conditions (or the embeddability conditions) (104). Figures 11, 12, and 13 are obtained by plotting $\boldsymbol{f}$ in $\mathbb{R}^3$ following the numerical integration of (148). We fix the rigid body motion of the surface by assuming $\boldsymbol{f}(0,0) = \mathbf{0}$, $\boldsymbol{f}_{,X}(0,0) = (\sqrt{G_{11}(0,0)}, 0, 0)^{\mathsf{T}}$, and $\boldsymbol{f}_{,Y}(0,0) = (0, \sqrt{G_{22}(0,0)}, 0)^{\mathsf{T}}$, where T denotes transpose of a vector in $\mathbb{R}^3$.

3.4.2 A morphoelastic circular shell

In the absence of body forces, we consider an initially planar thin morphoelastic circular disk $\mathcal{B}$ with vanishing boundary loads. We assume that the disk is undergoing radially-symmetric but non-uniform growth through its thickness such that the radial and circumferential curvatures are evolving while the intrinsic metric of the shell remains unchanged. Let (R, Φ, Z) be the standard cylindrical coordinate system for $\mathbb{R}^3$ such that initially the mid-surface $\mathcal{H}$ of the shell lies in the hyperplane $Z = 0$ and the origin of the coordinate system coincides with the center of the circular mid-surface. Let R_o be the radius of the circular disk. For time $t \geq 0$, we represent the aforementioned growth by the following evolving material metric:

$$\bar{\boldsymbol{G}} = \begin{pmatrix} e^{2\omega_R(R,Z,t)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R,Z,t)} R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{149}$$

such that ω_A for $A = R, \Theta$ are symmetric with respect to Z, i.e., $\omega_A(R,Z,t) = -\omega_A(R,-Z,t)$, which implies that $\omega_A(R,0,t) = 0$.[13] Therefore, the first and the second fundamental forms of $\mathcal{H}$ read

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} -K_R(R,t) & 0 \\ 0 & -R^2 K_\Theta(R,t) \end{pmatrix},$$

where $K_A(R,t) = \frac{\partial \omega_A}{\partial Z}(R,0,t)$ for $A = R, \Theta$.

[13] We let for example $\omega_A = Z K_A(R,t)$ for $A = R, \Theta$ in (149).

We endow the ambient Euclidean space $\mathbb{R}^3$ with the standard cylindrical coordinate system (r, ϕ, z) . In order to study the growth of the shell and obtain the growth-induced residual stresses, we embed the shell into the Euclidean ambient space and look for solutions of the form $(r, \phi, z) = (r(R,t), \Phi, z(R,t))$. We fix the rigid body motion of the embedded surface by assuming $r(0,t) = 0$, $z(0,t) = 0$, and $z'(0,t) = 0$. Therefore, the deformation tensors read

$$\boldsymbol{C} = \begin{pmatrix} r'^2 + z'^2 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \boldsymbol{\Theta} = \frac{1}{\left(r'^2 + z'^2\right)^{1/2}} \begin{pmatrix} r'z'' - r''z' & 0 \\ 0 & rz' \end{pmatrix},$$

where $r' = \frac{\partial r}{\partial R}$ and $r'' = \frac{\partial^2 r}{\partial R^2}$. We introduce the function $\chi = \chi(R,t)$ defined such that $z' = \chi r'$. Hence, the deformation tensors in terms of r and χ read

$$\boldsymbol{C} = \begin{pmatrix} r'^2\left(1+\chi^2\right) & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \boldsymbol{\Theta} = \frac{1}{\left(1+\chi^2\right)^{1/2}} \begin{pmatrix} \chi' r' & 0 \\ 0 & \chi r \end{pmatrix}.$$

We assume that the shell is made of a homogeneous and isotropic material. Because of the symmetry of the problem, the second Piola-Kirchhoff stress and material couple stress components have the following forms

$$\begin{gathered} S^{RR} = S^{RR}(R,t)\,, \quad S^{R\Phi} = 0\,, \quad S^{\Phi\Phi} = S^{\Phi\Phi}(R,t)\,, \\ M^{RR} = M^{RR}(R,t)\,, \quad M^{R\Phi} = 0\,, \quad M^{\Phi\Phi} = M^{\Phi\Phi}(R,t)\,. \end{gathered}$$

In the convected manifold $(\mathcal{H}, \boldsymbol{C})$, the only non-zero Christoffel symbols of the Levi-Civita connection are

$$\check{\Gamma}^R{}_{RR} = \frac{\chi\chi'}{1+\chi^2} + \frac{r''}{r'}\,, \quad \check{\Gamma}^R{}_{\Phi\Phi} = -\frac{r}{r'\left(1+\chi^2\right)}\,, \quad \check{\Gamma}^\Phi{}_{R\Phi} = \frac{r'}{r}\,.$$

Therefore, the equilibrium equations follow from (126) and read[14]

$$\begin{aligned}
&\frac{1}{R}\left[R\left(S^{RR}+2\frac{\chi'}{r'\left(1+\chi^2\right)^{3/2}}M^{RR}\right)\right]_{,R} \\
&+2\left(\frac{\chi\chi'}{1+\chi^2}+\frac{r''}{r'}\right)\left(S^{RR}+2\frac{\chi'}{r'\left(1+\chi^2\right)^{3/2}}M^{RR}\right) \\
&-\frac{r}{r'\left(1+\chi^2\right)}\left(S^{\Phi\Phi}+2\frac{\chi}{r\left(1+\chi^2\right)^{1/2}}M^{\Phi\Phi}\right)-\left(\frac{\chi'}{r'\left(1+\chi^2\right)^{3/2}}\right)_{,R}M^{RR} \\
&-\left(\frac{\chi'}{r'\left(1+\chi^2\right)^{3/2}}-\frac{\chi}{r\left(1+\chi^2\right)^{1/2}}\right)\frac{r}{r'\left(1+\chi^2\right)}M^{\Phi\Phi}=0\,,
\end{aligned}\tag{151a}$$

$$\begin{aligned}
&\frac{\chi' r'}{\left(1+\chi^2\right)^{1/2}}\left(S^{RR}+\frac{\chi'}{r'\left(1+\chi^2\right)^{3/2}}M^{RR}\right) \\
&+\frac{\chi r}{\left(1+\chi^2\right)^{1/2}}\left(S^{\Phi\Phi}+\frac{\chi}{r\left(1+\chi^2\right)^{1/2}}M^{\Phi\Phi}\right)-\frac{r}{R}\left(\frac{r'}{r}+\frac{\chi\chi'}{1+\chi^2}+\frac{r''}{r'}\right) \\
&\times\left[\frac{1}{r}\left(RM^{RR}\right)_{,R}+2\frac{R}{r}\left(\frac{\chi\chi'}{1+\chi^2}+\frac{r''}{r'}\right)M^{RR}-\frac{R}{r'\left(1+\chi^2\right)}M^{\Phi\Phi}\right] \\
&-\frac{r}{R}\left[\frac{1}{r}\left(RM^{RR}\right)_{,R}+2\frac{R}{r}\left(\frac{\chi\chi'}{1+\chi^2}+\frac{r''}{r'}\right)M^{RR}-\frac{R}{r'\left(1+\chi^2\right)}M^{\Phi\Phi}\right]_{,R}=0\,.
\end{aligned}\tag{151b}$$

The boundary conditions (127) for zero surface load, zero moment load, and zero shear load at $R=R_o$ read

$$S^{RR}=0\,,\quad M^{RR}=0\,,\quad \frac{1}{R}\left(RM^{RR}\right)'-rr'M^{\Phi\Phi}=0\,.\tag{152}$$

We assume a Saint Venant-Kirchhoff constitutive model, for which the strain energy density $\mathcal{W}$ is given by (136). Therefore, the non-zero components of the

[14]Note that $\left(\Sigma^{AB}+C^{-AC}\Theta_{CD}\Lambda^{DB}\right)_{||B}+C^{-AC}\Theta_{CD}\Lambda^{DB}{}_{||B}=\left(\Sigma^{AB}+2C^{-AC}\Theta_{CD}\Lambda^{DB}\right)_{||B}-\left(C^{-AC}\Theta_{CD}\right)_{||B}\Lambda^{DB}$. Also, we have $\mathrm{J}=\frac{r}{R}$. Therefore, the convected stress and couple stress tensors read $\boldsymbol{\Sigma}=\frac{R}{r}\boldsymbol{S}$ and $\boldsymbol{\Lambda}=\frac{R}{r}\boldsymbol{M}$.

material stress and couple stress tensors read

$$
\begin{aligned}
S^{RR} &= Y_c h \left\{ \left[r'^2 \left(1+\chi^2\right) - 1 \right] + \nu \left(\frac{r^2}{R^2} - 1 \right) \right\} , \\
S^{\Phi\Phi} &= Y_c h \left\{ \nu \left[r'^2 \left(1+\chi^2\right) - 1 \right] + \left(\frac{r^2}{R^2} - 1 \right) \right\} \frac{1}{R^2} , \\
M^{RR} &= Y_c h^3 \left\{ \left[\frac{\chi' r'}{\left(1+\chi^2\right)^{1/2}} + K_R \right] + \nu \left[\frac{\chi r}{R^2 \left(1+\chi^2\right)^{1/2}} + K_\Theta \right] \right\} , \\
M^{\Phi\Phi} &= Y_c h^3 \left\{ \nu \left[\frac{\chi' r'}{\left(1+\chi^2\right)^{1/2}} + K_R \right] + \left[\frac{\chi r}{R^2 \left(1+\chi^2\right)^{1/2}} + K_\Theta \right] \right\} \frac{1}{R^2} ,
\end{aligned}
$$

where $Y_c = \frac{E}{2(1-\nu^2)}$. Therefore, we have the governing equations (151) and the boundary conditions (152) in terms of r and χ . However, we still need the kinetic equations for the evolution of growth in order to fully solve the problem. We consider the Rayleigh potential $\mathcal{R}(\dot{\boldsymbol{B}}, \boldsymbol{G}) = \beta_1 \mathrm{tr}(\dot{\boldsymbol{B}}) + \beta_2 \mathrm{tr}(\dot{\boldsymbol{B}}^2)$. Therefore, we find the following evolution equations for the principal curvatures:

$$
\dot{K}_R = \frac{\beta_1}{2\beta_2} - \frac{Y_c h^3}{12\beta_2} \left\{ \left[\frac{\chi' r'}{\left(1+\chi^2\right)^{1/2}} + K_R \right] + \nu \left[\frac{\chi r}{R^2 \left(1+\chi^2\right)^{1/2}} + K_\Theta \right] \right\} , \tag{154a}
$$

$$
\dot{K}_\Theta = \frac{\beta_1}{2\beta_2} - \frac{Y_c h^3}{12\beta_2} \left\{ \nu \left[\frac{\chi' r'}{\left(1+\chi^2\right)^{1/2}} + K_R \right] + \left[\frac{\chi r}{R^2 \left(1+\chi^2\right)^{1/2}} + K_\Theta \right] \right\} . \tag{154b}
$$

We define the following characteristic time

$$
\tau = \frac{12\beta_2}{Y_c h^3} . \tag{155}
$$

Using the time differentiation with respect to $\hat{t} = t/\tau$ and the spatial differentiation with respect to $\hat{R} = R/R_o$, (154) reads

$$
\begin{aligned}
\dot{\hat{K}}_R &= \frac{6\beta_1 R_o}{Y_c h^3} - \left\{ \left[\frac{\chi' \hat{r}'}{\left(1+\chi^2\right)^{1/2}} + \hat{K}_R \right] + \nu \left[\frac{\chi \hat{r}}{\hat{R}^2 \left(1+\chi^2\right)^{1/2}} + \hat{K}_\Theta \right] \right\} , \\
\dot{\hat{K}}_\Theta &= \frac{6\beta_1 R_o}{Y_c h^3} - \left\{ \nu \left[\frac{\chi' \hat{r}'}{\left(1+\chi^2\right)^{1/2}} + \hat{K}_R \right] + \left[\frac{\chi \hat{r}}{\hat{R}^2 \left(1+\chi^2\right)^{1/2}} + \hat{K}_\Theta \right] \right\} ,
\end{aligned}
$$

where $\hat{K}_A = K_A R_o$ for $A = R, \Theta$, $\hat{r} = \frac{r}{R_o}$, $\hat{R} = \frac{R}{R_o}$, and $\hat{h} = \frac{h}{R_o}$.

We numerically solve the governing equations (151) and the boundary conditions (152) along with the kinetic equations (154) for a shell of thickness $h = 0.1 R_o$ made

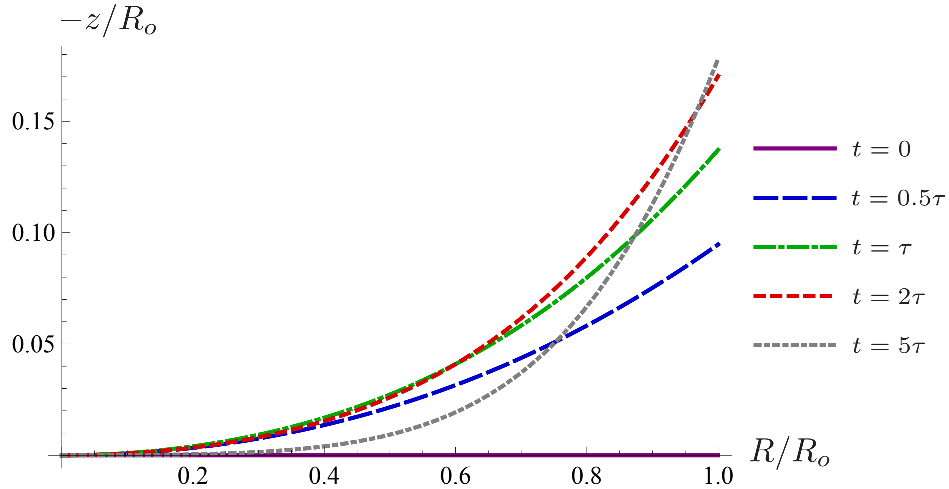


Figure 14: Evolution of (i) the spatial radius $r = r(R, t)$ and (ii) the out-of-plane spatial elevation profile $z = z(R, t)$ of a radial fiber of the initially planar growing disk.

of an isotropic and homogeneous material with $\nu = 0.5$, and undergoing a growth such that $\beta_1 = Y_c h^3/(12R_o)$. In Figure 14 we show the evolution of a radial fiber by plotting its radius r and elevation z as a function of R. We observe that the radius r remains almost unchanged from R and is almost time-independent while the out-of-plane elevation changes the fiber from its original configuration on the plane $z = 0$ to adopt a curved configuration. In Figure 15, we show the evolution of the spatial embedding of the disk from its initial planar configuration to a non-trivial curved disk. In Figures 16 and 17, we show the evolution of the radial and the circumferential curvatures. In Figures 18 and 19, we show the evolutions of the residual stresses and couple-stresses in the growing disk.

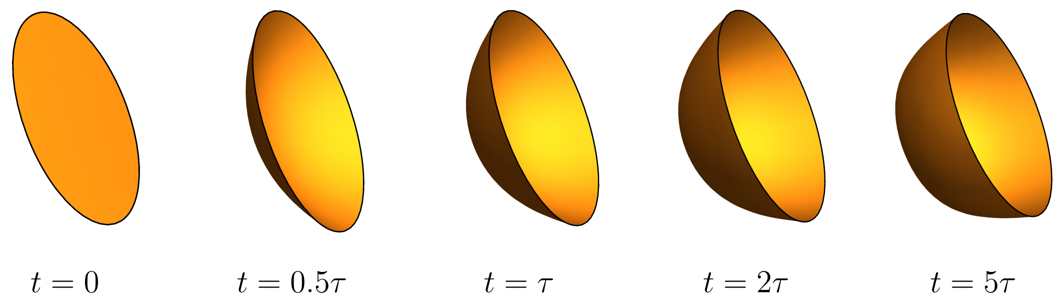


Figure 15: Visualization of the evolution of an initially planar growing disk. The out-of-plane elevation is scaled to a factor of 5.

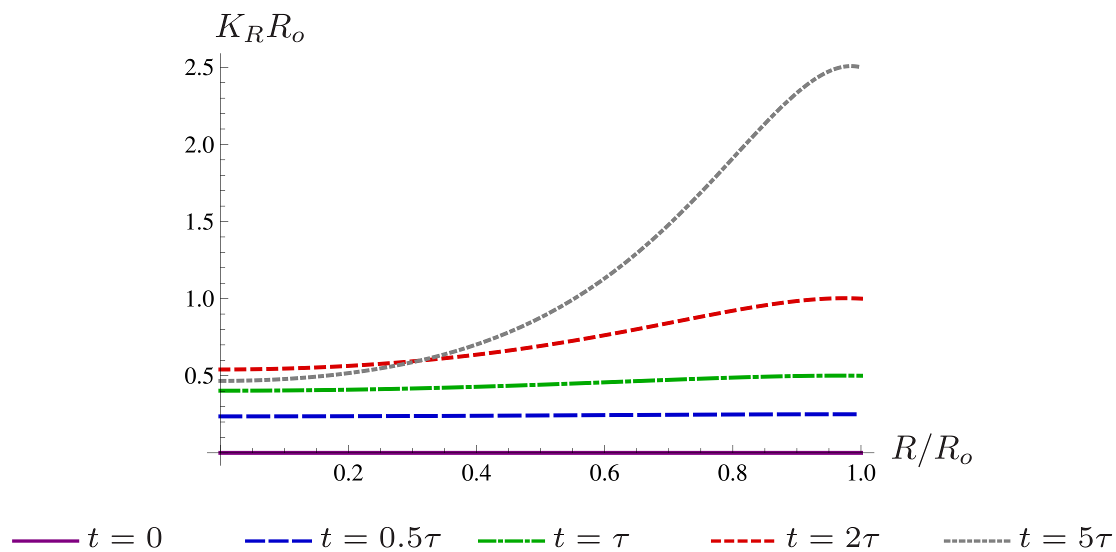


Figure 16: Evolution of the radial principal curvature on a radial fiber of the initially planar growing disk.

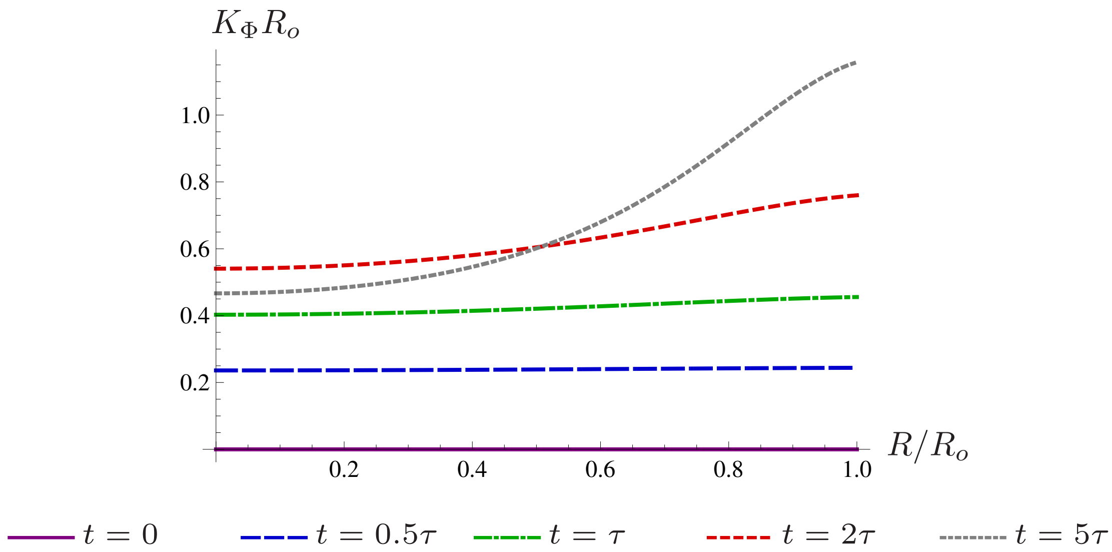


Figure 17: Evolution of the circumferential principal curvature on a radial fiber of the initially planar growing disk.

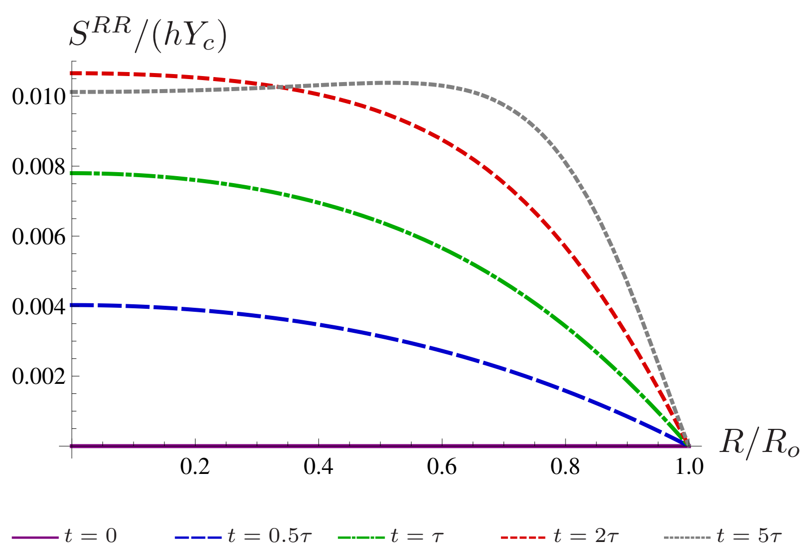


Figure 18: Evolution of the radial stress on a radial fiber of the initially planar growing disk.

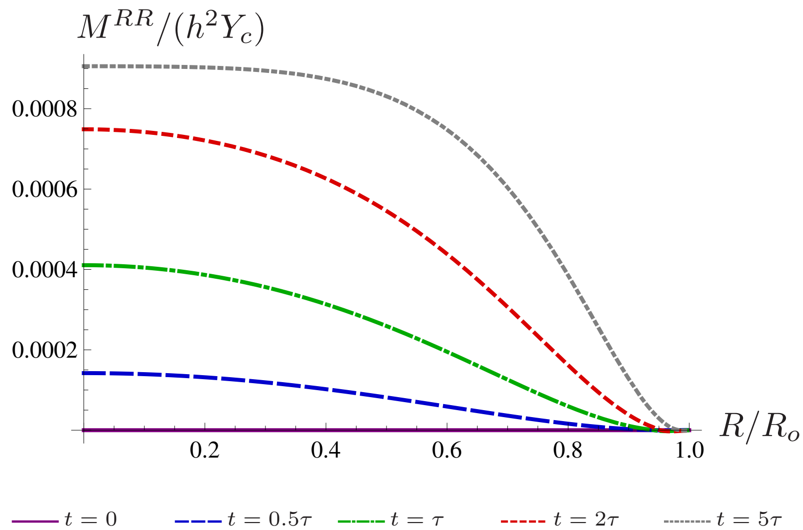


Figure 19: Evolution of the radial couple-stresses on a radial fiber of the initially planar growing disk.

CHAPTER IV

SMALL-ON-LARGE GEOMETRIC ANELASTICITY

The complexity of the equations of nonlinear elasticity, in particular for anelasticity, leaves little hope for exact solutions to be found. A few exact solutions can be found by semi-inverse methods assuming some symmetry-restrictive classes of deformations as it has been done in the previous sections of this work. As soon as this symmetry is broken, the governing equations start to be utterly complicated leaving no choice but for equally complicated numerical computations. As a sequel to the developments carried out in the previous sections toward a geometric theory of anelasticity, we consider in this section small perturbations superposed on a finite distribution of the source of anelasticity. In this framework, such perturbations are achieved by perturbing the referential geometry, i.e., perturbing the material metric in the context of three-dimensional elasticity, and perturbing the fundamental forms in the context of shell elasticity. The resulting governing equations are linear. Even in the case when one fails to find exact solutions, these are much easier to deal with numerically. If the perturbation lacks the symmetry of the original distribution, one hence generates new solutions for anelastic problems in the form of small elastic deformation superposed on the finite deformation. One could start from a field for the source of anelasticity/residual stresses with an existing equilibrium solution, perturb it and solve for the induced small elastic deformations due to the resulting material metric perturbation. Besides generating new solutions, this approach can also be used to examine the stability of the existing class of deformations used in the context of semi-inverse methods. In the following, we present the framework and the governing equations for the small elastic deformations due to a material metric perturbation in the context of three-dimensional elasticity. We supplement the theoretical developments with an application to screw dislocations and find indeed an exact solution for a non-symmetric distribution of such defects. Note that the results of this section have been previously reported in our published work [71].

4.1 An overview of nonlinear elasticity

We briefly review in the following some elements of the geometric formulation of anelasticity. Let $(\mathcal{B}, \boldsymbol{G})$ the stress-free material manifold. Let $(\mathcal{S}, \boldsymbol{g})$ be the Euclidean ambient space. We adopt the standard convention to denote objects and indices by uppercase characters in the material manifold $\mathcal{B}$ (e.g., $X \in \mathcal{B}$) and by lowercase characters in the spatial manifold $\mathcal{S}$ (e.g., $x \in \mathcal{S}$). Let $\nabla^{\boldsymbol{G}}$, and $\nabla^{\boldsymbol{g}}$ be the Levi-Civita connections of $(\mathcal{B}, \boldsymbol{G})$, and $(\mathcal{S}, \boldsymbol{g})$, repectively. We denote their respective Christoffel symbols by $\Gamma^A{}_{BC}$, and $\gamma^a{}_{bc}$, in the local coordinate charts $\{X^A\}$ and $\{x^a\}$, respectively.

Recall that the deformation gradient $\boldsymbol{F}$ is defined as $\boldsymbol{F}(X,t) := T\varphi_t(X)$, the Jacobian J is $J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F}$, and the right Cauchy-Green deformation tensor is $\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}}\boldsymbol{F}$. We also define the left Cauchy-Green deformation tensor (also called Finger tensor) as $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathsf{T}}$. In components, $b^a{}_b = F^a{}_A F^c{}_B G^{AB} g_{cb}$. Note that $\boldsymbol{b}^{-\flat}$ agrees with the push-forward of the material metric $\boldsymbol{G}$ by φ, i.e., $\boldsymbol{b}^{-\flat} = \varphi_* \boldsymbol{G}$, where $(.)^{-\flat}$ denotes the inverse operator followed by the flat operator. We define the convective manifold as the Riemannian manifold $(\mathcal{B}, \boldsymbol{C}^\flat)$. Let $\nabla^{\boldsymbol{C}}$ be the Levi-Civita connection of $(\mathcal{B}, \boldsymbol{C}^\flat)$. We denote its corresponding Christoffel symbols in the local coordinate chart $\{X^A\}$ by $\tilde{\Gamma}^A{}_{BC}$.

We denote the material and spatial mass densities by ρ_o and ρ, respectively. The conservation of mass in local form reads $\rho J = \rho_o$, which is equivalent to

$$\frac{d\rho}{dt} + \rho \operatorname{div}_{\boldsymbol{g}} \boldsymbol{v} = 0 \,,$$

where $\operatorname{div}_{\boldsymbol{g}}$ denotes the spatial divergence operator.

We assume that the body is made of a hyperelastic material, so that the constitutive model is given by an energy function $\mathcal{W}=\tilde{\mathcal{W}}(X,\boldsymbol{F},\boldsymbol{g},\boldsymbol{G})$[1] per unit undeformed volume, and the Cauchy stress tensor is given by [18]

$$\boldsymbol{\sigma}=\frac{2}{J}\frac{\partial\tilde{\mathcal{W}}}{\partial\boldsymbol{g}}\,, \tag{157}$$

which in components reads $\sigma^{ab}=\frac{2}{J}\frac{\partial\tilde{\mathcal{W}}}{\partial g_{ab}}$. We can alternatively consider $\mathcal{W}=\hat{\mathcal{W}}(X,\boldsymbol{C}^\flat,\boldsymbol{G})$ and the convected stress tensor $\boldsymbol{\Sigma}=\varphi_t^*\boldsymbol{\sigma}$ is written as [76]

$$\boldsymbol{\Sigma}=\frac{2}{J}\frac{\partial\hat{\mathcal{W}}}{\partial\boldsymbol{C}^\flat}\,, \tag{158}$$

which in components reads $\Sigma^{ab}=\frac{2}{J}\frac{\partial\hat{\mathcal{W}}}{\partial C_{AB}}$. If the material is incompressible, we have $J=1$ and the stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ are written as

$$\boldsymbol{\sigma}=2\frac{\partial\tilde{\mathcal{W}}}{\partial\boldsymbol{g}}-p\,\boldsymbol{g}^\sharp\,,\quad \boldsymbol{\Sigma}=2\frac{\partial\hat{\mathcal{W}}}{\partial\boldsymbol{C}^\flat}-p\,\boldsymbol{C}^{-\sharp}\,, \tag{159}$$

where p is the Lagrange multiplier associated with the incompressibility constraint, and $(.)^{-\sharp}$ denotes the inverse operator followed by the sharp operator for raising tensor indices. If the material is isotropic, the strain-energy function is expressed as a function of the principal invariants $I_1=\operatorname{tr}\boldsymbol{C}$, $I_2=\frac{1}{2}(\operatorname{tr}(\boldsymbol{C})^2-\operatorname{tr}(\boldsymbol{C}^2))$, and J, i.e., $\mathcal{W}=\bar{\mathcal{W}}(X,I_1,I_2,J)$, and the stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ can be written as [18, 84]

$$\boldsymbol{\sigma}=\left(\bar{\mathcal{W}}_J+\frac{2I_2}{J}\bar{\mathcal{W}}_{I_2}\right)\boldsymbol{g}^\sharp+\frac{2}{J}\bar{\mathcal{W}}_{I_1}\boldsymbol{b}^\sharp-2J\bar{\mathcal{W}}_{I_2}\boldsymbol{b}^{-\sharp}\,, \tag{160a}$$

$$\boldsymbol{\Sigma}=\frac{2}{J}\left(\bar{\mathcal{W}}_{I_1}+I_1\bar{\mathcal{W}}_{I_2}\right)\boldsymbol{G}^\sharp-\frac{2}{J}\bar{\mathcal{W}}_{I_2}\boldsymbol{C}^\sharp+\bar{\mathcal{W}}_J\boldsymbol{C}^{-\sharp}\,, \tag{160b}$$

where $\bar{\mathcal{W}}_{I_1}=\frac{\partial\bar{\mathcal{W}}}{\partial I_1}$, $\bar{\mathcal{W}}_{I_2}=\frac{\partial\bar{\mathcal{W}}}{\partial I_2}$, and $\bar{\mathcal{W}}_J=\frac{\partial\bar{\mathcal{W}}}{\partial J}$. If the material is incompressible and isotropic, one has

$$\boldsymbol{\sigma}=\left(2I_2\bar{\mathcal{W}}_{I_2}-p\right)\boldsymbol{g}^\sharp+2\bar{\mathcal{W}}_{I_1}\boldsymbol{b}^\sharp-2\bar{\mathcal{W}}_{I_2}\boldsymbol{b}^{-\sharp}\,, \tag{161a}$$

$$\boldsymbol{\Sigma}=2\left(\bar{\mathcal{W}}_{I_1}+I_1\bar{\mathcal{W}}_{I_2}\right)\boldsymbol{G}^\sharp-2\bar{\mathcal{W}}_{I_2}\boldsymbol{C}^\sharp-p\boldsymbol{C}^{-\sharp}\,. \tag{161b}$$

[1] The dependence of the energy function $\tilde{\mathcal{W}}$ on the metrics follows from the fact that $\tilde{\mathcal{W}}$ is a scalar that depends on the deformation gradient $\boldsymbol{F}$. This requires the metrics to obtain a scalar out of it, e.g., $\operatorname{tr}(\boldsymbol{F}^\mathsf{T}\boldsymbol{F})=F^a{}_AF^b{}_BG^{AB}g_{ab}$.

In spatial form, the balance of linear and angular momenta read

$$\operatorname{div}_{\boldsymbol{g}}\boldsymbol{\sigma} + \rho\mathsf{f} = \rho\boldsymbol{a}\,, \qquad \boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}\,, \tag{162}$$

where f denotes the body force per unit mass. The balance of linear and angular momenta in terms of the convected stress tensor read [76] (Note that, since $\nabla^{\boldsymbol{C}} = \varphi_t^*\nabla^{\boldsymbol{G}}$, the convective balance of momenta (163) can alternatively be obtained directly from the classical spatial balance of momenta (162).)

$$\operatorname{Div}_{\boldsymbol{C}}\boldsymbol{\Sigma} + \rho\varphi_t^*\mathsf{F} = \rho\varphi_t^*\boldsymbol{A}\,, \qquad \boldsymbol{S}^{\mathsf{T}} = \boldsymbol{S}\,, \tag{163}$$

where $\operatorname{Div}_{\boldsymbol{C}}$ denotes the divergence operator with respect to $\boldsymbol{C}^\flat$, and $\mathsf{F} := \mathsf{f}\circ\varphi_t$.

4.2 Small-on-Large Deformations Due to a Material Metric Perturbation

In this section, we formulate a theory of small superposed deformations due to a perturbation of the material metric. Given a motion φ_t with respect to a reference configuration $(\mathcal{B}, \boldsymbol{G})$, we consider a 1-parameter family of metrics $\boldsymbol{G}_\epsilon$ such that $\boldsymbol{G}_0 = \boldsymbol{G}$. We want to understand how the state of stress in the body is affected by such a perturbation. Note that a perturbation of the material metric is due to a perturbation of the source of anelasticity, e.g. a defect density. The variation of the material metric is defined as

$$\delta\boldsymbol{G} := \frac{d}{d\epsilon}\Big|_{\epsilon=0}\boldsymbol{G}_\epsilon\,.$$

For a small enough ϵ, one can write $\boldsymbol{G}_\epsilon = \boldsymbol{G} + \epsilon\delta\boldsymbol{G} + \boldsymbol{o}(\epsilon)$. Note that even though the deformation is seemingly independent of the material metric, changing the material metric may affect the equilibrium configuration of the body at any given time t. Hence a perturbation of the material metric may lead to a perturbation $\varphi_{t,\epsilon}$ of the motion, such that $\varphi_{t,0} = \varphi_t$ is the equilibrium configuration corresponding to the metric $\boldsymbol{G}_0 = \boldsymbol{G}$. We define its corresponding variation as

$$\delta\varphi_t(X) := \varphi_{t,X*}\partial_\epsilon\big|_{\epsilon=0} \in T_{\varphi_t(X)}\mathcal{S}\,,$$

that is, $\delta\varphi_t = \delta\varphi_t^a \partial_a$ and $\delta\varphi_t^a(X) := \frac{d\varphi_{X,t}}{d\epsilon}\big|_{\epsilon=0}$. Note that $\delta\varphi \circ \varphi^{-1}$ is the displacement field in the classical theory of linear elasticity and we denote it by $\boldsymbol{U} = \delta\varphi \circ \varphi^{-1}$. Since $\mathcal{S} = \mathbb{R}^3$, using the linear structure of $\mathbb{R}^3$, one can write for a small enough ϵ: $\varphi_\epsilon = \varphi + \epsilon\delta\varphi + o(\epsilon)$. Given the configuration φ resulting in the stress field $\boldsymbol{\sigma}$, the perturbed configuration φ_ϵ due to the material metric perturbation $\boldsymbol{G}_\epsilon$ induces a stress field, which for a small enough ϵ reads $\boldsymbol{\sigma}_\epsilon = \boldsymbol{\sigma} + \epsilon\delta\boldsymbol{\sigma} + o(\epsilon)$. In the following, we formulate the governing equations to solve for $\delta\varphi$ and find $\delta\boldsymbol{\sigma}$ is terms of $\delta\boldsymbol{G}$ and $\delta\varphi$.

As ϵ varies, for fixed X and t, the right Cauchy-Green tensor $\boldsymbol{C}_\epsilon^\flat$ remains in the same space $\mathcal{T}^2(T_X^*\mathcal{B})$, the set of $\binom{0}{2}$-rank tensors at X. Thus, it makes sense to define its variation as $\delta\boldsymbol{C}^\flat = \frac{d\boldsymbol{C}_\epsilon^\flat}{d\epsilon}\Big|_{\epsilon=0}$. One can write $\delta\boldsymbol{C}^\flat$ as follows

$$\delta\boldsymbol{C}^\flat = \frac{d}{d\epsilon}\boldsymbol{C}_\epsilon^\flat\Big|_{\epsilon=0} = \varphi_t^*\,\frac{d}{d\epsilon}\left[\varphi_{t*}\varphi_{t,\epsilon}^*\boldsymbol{g}\right]\Big|_{\epsilon=0} = \varphi_t^*\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{g} = \varphi_t^*\left(\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat + \left[\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat\right]^\mathsf{T}\right) = 2\varphi_t^*\boldsymbol{\epsilon}\,,$$

where $(.)^\mathsf{T}$ denotes the transpose operator, and $\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat + \left[\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat\right]^\mathsf{T}\right)$ is the linearized strain. The variation of the Jacobian of the motion reads[2]

$$\delta J = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\sqrt{\frac{\det\boldsymbol{C}_\epsilon^\flat}{\det\boldsymbol{G}}} = \left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp - \frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)J\,, \tag{164}$$

where ":" denotes the double contraction tensor product. Using $\rho J = \rho_o$ and the above equation (164), the variation of the spatial mass density reads

$$\delta\rho = -\left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp - \frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)\rho\,. \tag{165}$$

Note that when ϵ varies, the terms in the balance of linear momentum (163) are vectors that remain in the same vector space $T_X\mathcal{B}$.[3] Hence, one can write its variation as

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\mathrm{Div}_{\boldsymbol{C}_\epsilon}\boldsymbol{\Sigma}_\epsilon + \rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{B}\right] = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\left[\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\,,$$

[2]Recall that if $\det\boldsymbol{A}\neq 0$, one has $\frac{d\det\boldsymbol{A}}{d\boldsymbol{A}} = (\det\boldsymbol{A})\,\boldsymbol{A}^{-\mathsf{T}}$. Here, $\det\boldsymbol{G}\neq 0$ and $\det\boldsymbol{C}^\flat\neq 0$.

[3]However, note that when ϵ varies, the terms in the balance of linear momentum (162) are vectors that lie in the vector space $T_{\varphi_{t,\epsilon}(X)}\mathcal{S}$, in which the base point $\varphi_{t,\epsilon}(X)$ depends on ϵ.

which, by expanding the divergence term in local coordinates, transforms to read

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left[\left(\Sigma_\epsilon^{AB}{}_{,B}+\Sigma_\epsilon^{AK}\tilde{\Gamma}_\epsilon^B{}_{BK}+\Sigma_\epsilon^{BK}\tilde{\Gamma}_\epsilon^A{}_{BK}\right)\partial_A+\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{B}\right]=\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left[\rho_\epsilon\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]. \tag{166}$$

For different values of ϵ and fixed X and t, $\boldsymbol{\Sigma}_\epsilon$ lie in the same space $\mathcal{T}^2(T_X\mathcal{B})$. Hence, one can define $\delta\boldsymbol{\Sigma}=\frac{d\boldsymbol{\Sigma}_\epsilon}{d\epsilon}\big|_{\epsilon=0}$, which is computed in (167a) following (158). On the other hand, the variation of the Cauchy stress can be defined as the push-forward of that of the convected stress, i.e., $\delta\boldsymbol{\sigma}=\varphi_{t*}\delta\boldsymbol{\Sigma}$. Therefore, one finds

$$\delta\boldsymbol{\Sigma}=\frac{4}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{C}^\flat\partial\boldsymbol{C}^\flat}:\varphi_t^*\boldsymbol{\epsilon}+\frac{2}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{C}^\flat}:\delta\boldsymbol{G}-\left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)\boldsymbol{\Sigma}, \tag{167a}$$

$$\delta\boldsymbol{\sigma}=\frac{4}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{g}\partial\boldsymbol{g}}:\boldsymbol{\epsilon}+\frac{2}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{g}}:\delta\boldsymbol{G}-\left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)\boldsymbol{\sigma}. \tag{167b}$$

We define the following fourth order elasticity tensors:

$$\mathbb{C}:=\frac{4}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{g}\partial\boldsymbol{g}},\quad \mathbb{D}:=\frac{2}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{g}}, \tag{168}$$

which in components read $\mathbb{C}^{abcd}=\frac{4}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial g_{ab}\partial g_{cd}}$, and $\mathbb{D}^{abAB}=\frac{2}{J}\frac{\partial^2\tilde{\mathcal{W}}}{\partial G_{AB}\partial g_{ab}}$. Using (163), (165) and (167a), the governing equation (166) for the incremental stress transforms to[4]

$$\begin{aligned}
&\mathrm{Div}_{\boldsymbol{C}}\left(\frac{4}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{C}^\flat\partial\boldsymbol{C}^\flat}:\varphi_t^*\boldsymbol{\epsilon}+\frac{2}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{C}^\flat}:\delta\boldsymbol{G}\right)-d_{\mathcal{B}}\left(\boldsymbol{\epsilon}:\boldsymbol{g}^\sharp-\frac{1}{2}\delta\boldsymbol{G}:\boldsymbol{G}^\sharp\right)\cdot\boldsymbol{\Sigma}\\
&\quad-2\Sigma^{BK}C^{-AL}\,\varphi_t^*\boldsymbol{\epsilon}|_{LM}\,\tilde{\Gamma}^M{}_{BK}\partial_A+\Sigma^{BK}C^{-AL}\left[\varphi_t^*\boldsymbol{\epsilon}|_{BL,K}+\varphi_t^*\boldsymbol{\epsilon}|_{KL,B}-\varphi_t^*\boldsymbol{\epsilon}|_{BK,L}\right]\partial_A\\
&\quad-2\Sigma^{AK}C^{-BL}\,\varphi_t^*\boldsymbol{\epsilon}|_{LM}\,\tilde{\Gamma}^M{}_{BK}\partial_A+\Sigma^{AK}C^{-BL}\left[\varphi_t^*\boldsymbol{\epsilon}|_{BL,K}+\varphi_t^*\boldsymbol{\epsilon}|_{KL,B}-\varphi_t^*\boldsymbol{\epsilon}|_{BK,L}\right]\partial_A\\
&\quad+\rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{B}\right]\bigg|_{\epsilon=0}=\rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\bigg|_{\epsilon=0},
\end{aligned} \tag{169}$$

where $d_{\mathcal{B}}$ denotes the exterior derivative operator on $\mathcal{B}$, i.e., for a function $f:\mathcal{B}\to\mathbb{R}$, one has $d_{\mathcal{B}}f=\frac{\partial f}{\partial X^A}dX^A$. Denoting by a double stroke $(.)_{||}$ the convective covariant

[4] Recall that the Christoffel symbols for the convective Levi-Civita connection, i.e., the Levi-Civita connection for the convective manifold $(\mathcal{B},\boldsymbol{C}^\flat)$, read $\tilde{\Gamma}^A{}_{BK}=\frac{1}{2}C^{-AL}\left(C_{BL,K}+C_{KL,B}+C_{BK,L}\right)$.

derivative, i.e., the covariant derivative in the convective manifold $(\mathcal{B}, \boldsymbol{C})$, one can write $(\varphi_t^*\boldsymbol{\epsilon})_{BL,K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL,B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK,L} = (\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK||L} + 2\,(\varphi_t^*\boldsymbol{\epsilon})_{LM}\,\tilde{\Gamma}^M{}_{BK}$. One can also show that

$$
\begin{aligned}
&C^{-BL}\left[(\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK||L}\right] \\
&= \left[C^{-BL}\,(\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + C^{-BL}\,(\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - C^{-BL}\,(\varphi_t^*\boldsymbol{\epsilon})_{BK||L}\right] \\
&= \left[\left(\boldsymbol{C}^{-1}\!:\!\varphi_t^*\boldsymbol{\epsilon}\right)_{,K} + C^{-IJ}\,(\varphi_t^*\boldsymbol{\epsilon})_{KJ||I} - C^{-JI}\,(\varphi_t^*\boldsymbol{\epsilon})_{JK||I}\right] = \left(\boldsymbol{g}^\sharp\!:\!\boldsymbol{\epsilon}\right)_{,K}\,.
\end{aligned}
$$

On the other hand, one has $d_{\mathcal{B}}\left(\boldsymbol{\epsilon}\!:\!\boldsymbol{g}^\sharp\right)\cdot\boldsymbol{\Sigma} = \left(\boldsymbol{g}^\sharp\!:\!\boldsymbol{\epsilon}\right)_{,K}\Sigma^{KA}\partial_A$. Therefore, (169) is simplified to read

$$
\begin{aligned}
&\mathrm{Div}_{\boldsymbol{C}}\left(\frac{4}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{C}^\flat\partial\boldsymbol{C}^\flat}\!:\!\varphi_t^*\boldsymbol{\epsilon} + \frac{2}{J}\frac{\partial^2\hat{\mathcal{W}}}{\partial\boldsymbol{G}\partial\boldsymbol{C}^\flat}\!:\!\delta\boldsymbol{G}\right) + d_{\mathcal{B}}\left(\frac{1}{2}\delta\boldsymbol{G}\!:\!\boldsymbol{G}^\sharp\right)\cdot\boldsymbol{\Sigma} \\
&+\Sigma^{BK}C^{-AL}\left[\varphi_t^*\boldsymbol{\epsilon}|_{BL||K} + \varphi_t^*\boldsymbol{\epsilon}|_{KL||B} - \varphi_t^*\boldsymbol{\epsilon}|_{BK||L}\right]\partial_A + \rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{B}\right]\Big|_{\epsilon=0} = \rho\,\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\Big|_{\epsilon=0}\,.
\end{aligned}
\tag{170}
$$

Recall that, $\nabla^{\boldsymbol{C}} = \varphi_t^*\nabla^{\boldsymbol{g}}$. Thus, one can write

$$
(\varphi_t^*\boldsymbol{\epsilon})_{AB||C} = F^a{}_A F^b{}_B F^c{}_C \epsilon_{ab|c} = \frac{1}{2}F^a{}_A F^b{}_B F^c{}_C\left(U_{a|bc} + U_{b|ac}\right)\,.
$$

Assuming that the ambient space is flat, it follows that $U_{a|bc} = U_{a|cb}$. Hence, it is straightforward to show that $(\varphi_t^*\boldsymbol{\epsilon})_{BL||K} + (\varphi_t^*\boldsymbol{\epsilon})_{KL||B} - (\varphi_t^*\boldsymbol{\epsilon})_{BK||L} = F^b{}_B F^k{}_K F^l{}_L U_{l|bk}$. For the acceleration vector, one has

$$
\begin{aligned}
\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{A}_\epsilon\right]\Big|_{\epsilon=0} &= \varphi_t^*\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{A} = \varphi_t^*\left[\frac{\partial A_\epsilon^a}{\partial\epsilon}\Big|_{\epsilon=0}\partial_a + \nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{A} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right] = \varphi_t^*\left[D^{\boldsymbol{g}}_\epsilon\boldsymbol{A} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right] \\
&= \varphi_t^*\left[D^{\boldsymbol{g}}_\epsilon D^{\boldsymbol{g}}_t\boldsymbol{V} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right] = \varphi_t^*\left[D^{\boldsymbol{g}}_t D^{\boldsymbol{g}}_\epsilon\boldsymbol{V} + \nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right] \\
&= \varphi_t^*\left[D^{\boldsymbol{g}}_t D^{\boldsymbol{g}}_t\boldsymbol{U} + \nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V} - \nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right]\,,
\end{aligned}
$$

where $D^{\boldsymbol{g}}_\epsilon$ denotes the covariant derivative along $\epsilon \to \varphi_{\epsilon,t}(X)$, for X and t fixed, and where we used $D^{\boldsymbol{g}}_\epsilon D^{\boldsymbol{g}}_t\boldsymbol{V} = D^{\boldsymbol{g}}_t D^{\boldsymbol{g}}_\epsilon\boldsymbol{V} + \nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V}$, since we assume a flat ambient space. We also use the symmetry lemma [56] to write $D^{\boldsymbol{g}}_\epsilon\boldsymbol{V} = D^{\boldsymbol{g}}_t\boldsymbol{U}$. For the body force vector, one similarly has $\frac{d}{d\epsilon}\left[\varphi_{\epsilon,t}^*\boldsymbol{B}\right]\big|_{\epsilon=0} = \varphi_t^*\boldsymbol{L}_{\boldsymbol{U}}\boldsymbol{B} = \varphi_t^*\left[\nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{B} - \nabla^{\boldsymbol{g}}_{\boldsymbol{B}}\boldsymbol{U}\right]$. Finally, using the above results and pushing forward (170) by φ_t, one obtains the following balance of linear momentum for the perturbed motion

$$\begin{aligned}\operatorname{div}_{\boldsymbol{g}}(\mathbb{C}\!:\!\boldsymbol{\epsilon}+\mathbb{D}\!:\!\delta\boldsymbol{G})+\varphi_{t*}d_{\mathcal{B}}\left(\frac{1}{2}\delta\boldsymbol{G}\!:\!\boldsymbol{G}^{\sharp}\right)\cdot\boldsymbol{\sigma}\\+\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}\!:\!\boldsymbol{\sigma}+\rho\left(\nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{B}-\nabla^{\boldsymbol{g}}_{\boldsymbol{B}}\boldsymbol{U}\right)=\rho\left(D^{\boldsymbol{g}}_tD^{\boldsymbol{g}}_t\boldsymbol{U}+\nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V}-\nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right),\end{aligned}\tag{171}$$

where $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}\!:\!\boldsymbol{\sigma}=\sigma^{ab}\nabla^{\boldsymbol{g}}_{\partial_a}\nabla^{\boldsymbol{g}}_{\partial_b}\boldsymbol{U}=\sigma^{ab}U^c{}_{|ba}\partial_c$. If the material is incompressible, the variation of the convected and the Cauchy stress tensors are written as

$$\delta\boldsymbol{\Sigma}=\varphi_t^*(\mathbb{C}\!:\!\boldsymbol{\epsilon}+\mathbb{D}\!:\!\delta)-\delta p\,\boldsymbol{C}^{-\sharp}+2p\,\varphi_t^*\boldsymbol{\epsilon}^{\sharp},\tag{172a}$$

$$\delta\boldsymbol{\sigma}=\mathbb{C}\!:\!\boldsymbol{\epsilon}+\mathbb{D}\!:\!\delta\boldsymbol{G}-\delta p\,\boldsymbol{g}^{\sharp}+2p\boldsymbol{\epsilon}^{\sharp},\tag{172b}$$

where $\delta p=\frac{d}{d\epsilon}\Big|_{\epsilon=0}p_\epsilon$ is the resulting pressure variation, which can also be interpreted as the Lagrange multiplier associated with the constraint $\delta J=0$. Therefore, for an incompressible solid, the balance of linear momentum for the perturbed motion reads

$$\begin{aligned}\operatorname{div}_{\boldsymbol{g}}\delta\boldsymbol{\sigma}+\varphi_{t*}d_{\mathcal{B}}\left(\frac{1}{2}\delta\boldsymbol{G}\!:\!\boldsymbol{G}^{\sharp}\right)\cdot\boldsymbol{\sigma}+\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}\!:\!\boldsymbol{\sigma}+\rho\left(\nabla^{\boldsymbol{g}}_{\boldsymbol{U}}\boldsymbol{B}-\nabla^{\boldsymbol{g}}_{\boldsymbol{B}}\boldsymbol{U}\right)\\=\rho\left(D^{\boldsymbol{g}}_tD^{\boldsymbol{g}}_t\boldsymbol{U}+\nabla^{\boldsymbol{g}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{V}-\nabla^{\boldsymbol{g}}_{\boldsymbol{A}}\boldsymbol{U}\right).\end{aligned}\tag{173}$$

Remark 4.2.1. Note that for an isotropic solid, one can show that the components of the elasticity tensors (168) read

$$
\begin{aligned}
\mathbb{C}^{abcd} &= \left(\bar{\mathcal{W}}_J+J\bar{\mathcal{W}}_{JJ}+4I_2\bar{\mathcal{W}}_{I_2J}+\frac{4I_2}{J}\bar{\mathcal{W}}_{I_2}+\frac{4I_2^2}{J}\bar{\mathcal{W}}_{I_2I_2}\right)g^{ab}g^{cd}+\frac{4}{J}\bar{\mathcal{W}}_{I_1I_1}b^{ab}b^{cd}\\
&-\left(\bar{\mathcal{W}}_J+\frac{2I_2}{J}\bar{\mathcal{W}}_{I_2}\right)\left(g^{ac}g^{bd}+g^{ad}g^{bc}\right)+\left(2\bar{\mathcal{W}}_{I_1J}+\frac{4I_2}{J}\bar{\mathcal{W}}_{I_1I_2}\right)\left(g^{ab}b^{cd}+b^{ab}g^{cd}\right)\\
&-J\left(2J\bar{\mathcal{W}}_{I_2J}+4\bar{\mathcal{W}}_{I_2}+4I_2\bar{\mathcal{W}}_{I_2I_2}\right)\left(g^{ab}b^{-cd}+b^{-ab}g^{cd}\right)+4J^3\bar{\mathcal{W}}_{I_2I_2}b^{-ab}b^{-cd}\\
&+2J\bar{\mathcal{W}}_{I_2}\left(b^{-ac}g^{bd}+b^{-ad}g^{bc}+b^{-bc}g^{ad}+b^{-bd}g^{ac}\right)-4J\bar{\mathcal{W}}_{I_1I_2}\left(b^{-ab}b^{cd}+b^{ab}b^{-cd}\right),
\end{aligned}
\tag{174a}
$$

$$
\begin{aligned}
\mathbb{D}^{abAB} &= -\left(\frac{1}{2}\bar{\mathcal{W}}_J+\frac{J}{2}\bar{\mathcal{W}}_{JJ}+2I_2\bar{\mathcal{W}}_{I_2J}+\frac{2I_2}{J}\bar{\mathcal{W}}_{I_2}+\frac{2I_2^2}{J}\bar{\mathcal{W}}_{I_2I_2}\right)g^{ab}G^{AB}\\
&-\frac{1}{J}\bar{\mathcal{W}}_{I_1}\left(F^a{}_KF^b{}_LG^{AK}G^{BL}+F^b{}_KF^a{}_LG^{AK}G^{BL}\right)-2J^3\bar{\mathcal{W}}_{I_2I_2}b^{-ab}C^{-AB}\\
&+J\left(J\bar{\mathcal{W}}_{I_2J}+2\bar{\mathcal{W}}_{I_2}+2I_2\bar{\mathcal{W}}_{I_2I_2}\right)\left(b^{-ab}G^{AB}+g^{ab}C^{-AB}\right)-\frac{2}{J}\bar{\mathcal{W}}_{I_1I_1}b^{ab}C^{AB}\\
&+2J\bar{\mathcal{W}}_{I_1I_2}\left(b^{-ab}C^{AB}+b^{ab}C^{-AB}\right)-\left(\bar{\mathcal{W}}_{I_1J}+\frac{2I_2}{J}\bar{\mathcal{W}}_{I_1I_2}\right)\left(b^{ab}G^{AB}+g^{ab}C^{AB}\right)\\
&-J\bar{\mathcal{W}}_{I_2}\left(g^{ak}g^{bl}F^{-A}{}_kF^{-B}{}_l+g^{bk}g^{al}F^{-A}{}_kF^{-B}{}_l\right).
\end{aligned}
\tag{174b}
$$

For an incompressible isotropic solid, the components of the elasticity tensors can be obtained from (174) by setting $J=1$ and removing the terms containing $\bar{\mathcal{W}}_J$.

4.3 Examples of Material Metric Perturbations in an Infinitely Long Cylindrical Bar with an Axi-Symmetric Distribution of Parallel Screw Dislocations

In this section, we solve examples of perturbed dislocation distributions. Starting from a dislocation distribution with an existing equilibrium solution, we perturb it and solve for the induced small elastic deformations due to the resulting material metric perturbation. We consider the example of a cylindrically-symmetric distribution of parallel screw dislocations in a cylinder made of an incompressible, isotropic, and radially inhomogeneous nonlinear elastic solid, i.e., a solid with an energy function that can be written as $\mathcal{W}=\bar{\mathcal{W}}(R,I_1,I_2)$. Using the geometric theory of nonlinear dislocation mechanics introduced in [90], we first construct the stress-free Weitzenböck material manifold for an arbitrary cylindrically-symmetric parallel screw-dislocations distribution. Next, considering a perturbation of the axi-symmetric dislocation distribution following §4.2, we solve for the induced small elastic deformations and the corresponding stress field.

4.3.1 Material metric perturbation

In a cylindrical coordinate system (R, Θ, Z), we consider a distribution of cylindrically-symmetric screw dislocations parallel to the Z-axis by assuming a Z-oriented radially-symmetric Burgers' vector density $b = b(R)$. Let us consider a perturbation of this Burgers' vector distribution, i.e., we take a one-parameter family of Burgers' vectors $b_\epsilon(R, \Theta, Z)$ such that $b_0(R, \Theta, Z) = b(R)$. We define its variation as $\delta b = \frac{d}{d\epsilon} b_\epsilon \big|_{\epsilon=0}$. The given distribution of Burgers' vectors is equivalent to having the following torsion 2-forms

$$\mathcal{T}^1 = \mathcal{T}^2 = 0\,, \quad \mathcal{T}^3_\epsilon = \frac{b_\epsilon(R, \Theta, Z)}{2\pi} \vartheta^1 \wedge \vartheta^2\,.$$

Following the method of Cartan's moving frames [80], we look for an orthonormal coframe field of the form $\vartheta^1 = dR,\ \vartheta^2 = R d\Theta,\ \vartheta^3 = dZ + f_\epsilon(R, \Theta, Z) d\Theta$, for some function $f_\epsilon = f_\epsilon(R, \Theta, Z)$ to be determined. Denoting by $\omega^\alpha{}_\beta$ the connection 1-forms, Cartan's first structural equations, $\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$, for $\alpha = 1, 2, 3$, give one the following non-zero connection coefficients

$$\omega^1{}_{22} = -\frac{1}{R}\,, \quad \omega^1{}_{32} = -\frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\,, \quad \omega^2{}_{13} = \omega^3{}_{21} = \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\,, \quad \omega^2{}_{33} = \frac{f_{\epsilon,Z}}{R}\,.$$

Hence, the connection 1-forms read

$$\omega^1{}_2 = -\frac{1}{R}\vartheta^2 - \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\vartheta^3,\ \omega^2{}_3 = \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\vartheta^1 + \frac{f_{\epsilon,Z}}{R}\vartheta^3,\ \omega^3{}_1 = \frac{1}{2}\left(\frac{f_{\epsilon,R}}{R} - \frac{b_\epsilon}{2\pi}\right)\vartheta^2.$$

Cartan's second structural equations, $\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge^\gamma{}_\beta$, for $\alpha, \beta = 1, 2, 3$, along with the flatness of the material manifold yield, [5] $f_{\epsilon,R} = R\frac{b_\epsilon}{2\pi}$, $f_{\epsilon,Z} = 0$.

[5]For dislocations, the material manifold is by construction a Weitzenböck manifold, i.e., it is flat and has a compatible connection with a possibly non-zero torsion [63].

Therefore, $b_{\epsilon,Z} = 0$, and hence $b_\epsilon = b_\epsilon(R,\Theta)$, i.e., a Z-dependent Burgers' vector cannot be accommodated using the assumed coframe field. It then follows that $f_\epsilon(R,\Theta) = \frac{1}{2\pi}\int_0^R \xi b_\epsilon(\xi,\Theta)d\xi$, and the perturbed material metric in the coordinate frame is written as

$$\boldsymbol{G}_\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f_\epsilon^2(R,\Theta) & f_\epsilon(R,\Theta) \\ 0 & f_\epsilon(R,\Theta) & 1 \end{pmatrix}.$$

Hence, the variation of the material metric is written as

$$\delta\boldsymbol{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2f(R)\delta f(R,\Theta) & \delta f(R,\Theta) \\ 0 & \delta f(R,\Theta) & 0 \end{pmatrix},$$

where

$$f(R) = \frac{1}{2\pi}\int_0^R \xi b(\xi)d\xi \qquad \text{and} \qquad \delta f(R,\Theta) = \frac{1}{2\pi}\int_0^R \xi \delta b(\xi,\Theta)d\xi\,.$$

Knowing that $b_0 = b(R)$, we have $f_0 = f(R) = \frac{1}{2\pi}\int_0^R \xi b(\xi)d\xi$ and $\boldsymbol{G}_0 = \boldsymbol{G}(R)$ is the metric for the axi-symmetric parallel screw dislocations

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f^2(R) & f(R) \\ 0 & f(R) & 1 \end{pmatrix}.$$

Note that $\operatorname{tr}(\delta\boldsymbol{G}) = \delta\boldsymbol{G}\!:\!\boldsymbol{G}^\sharp = 0$.

4.3.2 Stress perturbation

Let us first find the residual stress field for the finite axi-symmetric distribution assuming an incompressible isotropic solid. Based on the symmetry of the problem, we look for an embedding of the material manifold in the Euclidean ambient space such that, in cylindrical coordinates (r,θ,z), we have $\varphi(R,\Theta,Z) = (r(R),\Theta,Z)$. Then, the deformation gradient reads $\boldsymbol{F} = \operatorname{diag}(r'(R),1,1)$ and the Jacobian is written as

$J = rr'/R$. Using the incompressibility condition, i.e., $J = 1$, and assuming that $r(0) = 0$ to fix the rigid body translation of the body, we find that $r(R) = R$. Hence, the standard Euclidean metric for $\mathcal{S} = \mathbb{R}^3$ in cylindrical coordinates (r, θ, z) reads $\boldsymbol{g} = \mathrm{diag}(1, R^2, 1)$ and the only non-zero Christoffel symbols are $\gamma^r{}_{\theta\theta} = -R$ and $\gamma^\theta{}_{r\theta} = \frac{1}{R}$. The Finger deformation tensor is written as

$$\boldsymbol{b}^\sharp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & -\frac{f(R)}{R^2} \\ 0 & -\frac{f(R)}{R^2} & 1 + \frac{f^2(R)}{R^2} \end{pmatrix}.$$

Following (161a) and denoting $\alpha(R) = 2\bar{\mathcal{W}}_{I_1}(R, I_1(R), I_2(R))$ and $\beta(R) = 2\bar{\mathcal{W}}_{I_2}(R, I_1(R), I_2(R))$, the non-zero Cauchy stress components read

$$\sigma^{rr} = -p(R, \Theta, Z) + \alpha(R) + \left(\frac{f^2(R)}{R^2} + 2\right)\beta(R), \quad \sigma^{\theta\theta} = \frac{1}{R^2}\left[-p(R, \Theta, Z) + \alpha(R) + 2\beta(R)\right],$$
$$\sigma^{zz} = -p(R, \Theta, Z) + \left(\frac{f^2(R)}{R^2} + 1\right)\alpha(R) + \left(\frac{f^2(R)}{R^2} + 2\right)\beta(R), \quad \sigma^{\theta z} = -\frac{f(R)}{R^2}\left[\alpha(R) + \beta(R)\right].$$

Note that $I_1(R) = I_2(R) = 3 + f^2(R)/R^2$. The θ and z-equilibrium equations imply that $p = p(R)$, and the radial equilibrium equation is simplified to read $\sigma^{rr}{}_{,R} + \frac{1}{R}\sigma^{rr} - R\sigma^{\theta\theta} = 0$. Assuming a traction-free boundary condition on the boundary of the cylinder at $R = R_o$, we solve the above equation for $p = p(R)$ and it follows that the non-zero Cauchy stress components are

$$\begin{aligned} \sigma^{rr} &= \int_R^{R_o} \frac{f^2(\xi)}{\xi^3}\beta(\xi)d\xi, \quad \sigma^{\theta\theta} = \frac{1}{R^2}\left[\int_R^{R_o} \frac{f^2(\xi)}{\xi^3}\beta(\xi)d\xi - \frac{f^2(R)}{R^2}\beta(R)\right], \\ \sigma^{zz} &= \int_R^{R_o} \frac{f^2(\xi)}{\xi^3}\beta(\xi)d\xi + \frac{f^2(R)}{R^2}\alpha(R), \quad \sigma^{\theta z} = -\frac{f(R)}{R^2}\left[\alpha(R) + \beta(R)\right]. \end{aligned} \tag{175}$$

Next we formulate the governing equations for superposed small elastic deformation and compute the incremental deformation and residual stresses due to the perturbation δb. In cylindrical coordinates (r, θ, z), we look for solutions of the form $\delta\varphi(R, \Theta) = \boldsymbol{U}(R, \Theta) = (\delta r(R, \Theta), \delta\theta(R, \Theta), \delta z(R, \Theta))$. Hence, $\nabla^{\boldsymbol{g}}\boldsymbol{U}$ reads

$$U^a{}_{|b} = \begin{pmatrix} \delta r_{,R} & \delta r_{,\Theta} - R\delta\theta & 0 \\ \delta\theta_{,R} + \frac{\delta\theta}{R} & \delta\theta_{,\Theta} + \frac{\delta r}{R} & 0 \\ \delta z_{,R} & \delta z_{,\Theta} & 0 \end{pmatrix}.$$

Recalling that the linearized strain reads $\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat + \left[\nabla^{\boldsymbol{g}}\boldsymbol{U}^\flat\right]^{\mathsf{T}}\right)$, one can write

$$\boldsymbol{\epsilon} = \begin{pmatrix} \delta r_{,R} & \frac{1}{2}\left(\delta r_{,\Theta} + R^2 \delta\theta_{,R}\right) & \frac{1}{2}\delta z_{,R} \\ \frac{1}{2}\left(\delta r_{,\Theta} + R^2 \delta\theta_{,R}\right) & R^2\left(\delta\theta_{,\Theta} + \frac{1}{R}\delta r\right) & \frac{1}{2}\delta z_{,\Theta} \\ \frac{1}{2}\delta z_{,R} & \frac{1}{2}\delta z_{,\Theta} & 0 \end{pmatrix}.$$

Note that $\delta\boldsymbol{G}:\boldsymbol{G}^\sharp = 0$, and hence, the incompressibility condition $\delta J = 0$ using (164) is simplified to read

$$\frac{1}{R}\left(R\delta r\right)_{,R} + \delta\theta_{,\Theta} = 0\,. \tag{176}$$

In the absence of body forces, the equilibrium equation (173) simplifies to read

$$\mathrm{div}_{\boldsymbol{g}}\delta\boldsymbol{\sigma} + \nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}:\boldsymbol{\sigma} = \boldsymbol{0}\,, \tag{177}$$

where we recall that $\delta\boldsymbol{\sigma} = \left(\mathbb{C}:\boldsymbol{\epsilon} + \mathbb{D}:\delta\boldsymbol{G} - \delta p\, \boldsymbol{g}^\sharp + 2p\, \boldsymbol{\epsilon}^\sharp\right)$, $\delta p = \delta p(R,\Theta)$ is the Lagrange multiplier associated with the incompressibility condition $\delta J = 0$ (176), and $p = p(R)$ is the Lagrange multiplier associated with the incompressibility condition $J = 1$. Note that $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U}$ can be written in local coordinates as

$$U^a{}_{|bc} = \begin{pmatrix} \begin{pmatrix} \delta r_{,RR} \\ -R\delta\theta_{,R} - \frac{\delta r_{,\Theta}}{R} + \delta r_{,R\Theta} \\ 0 \end{pmatrix} & \begin{pmatrix} -R\delta\theta_{,R} - \frac{\delta r_{,\Theta}}{R} + \delta r_{,R\Theta} \\ -2R\delta\theta_{,\Theta} + \delta r_{,\Theta\Theta} + R\delta r_{,R} - \delta r \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{2\delta\theta_{,R}}{R} + \delta\theta_{,RR} \\ \frac{R\delta r_{,R} - \delta r}{R^2} + \delta\theta_{,R\Theta} \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{R\delta r_{,R} - \delta r}{R^2} + \delta\theta_{,R\Theta} \\ \delta\theta_{,\Theta\Theta} + R\delta\theta_{,R} + \frac{2\delta r_{,\Theta}}{R} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \delta z_{,RR} \\ -\frac{\delta z_{,\Theta}}{R} + \delta z_{,R\Theta} \\ 0 \end{pmatrix} & \begin{pmatrix} -\frac{\delta z_{,\Theta}}{R} + \delta z_{,R\Theta} \\ \delta z_{,\Theta\Theta} + R\delta z_{,R} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}. \tag{178}$$

For the sake of simplifying the calculations, let us assume that the body is made of a generalized neo-Hookean solid, i.e., the energy function has the form $\mathcal{W} = \bar{\mathcal{W}}(I_1)$.

Hence, it follows from (175) that the Cauchy stress reads

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\frac{f(R)}{R^2}\bar{\mathcal{W}}_{I_1} \\ 0 & -2\frac{f(R)}{R^2}\bar{\mathcal{W}}_{I_1} & 2\frac{f^2(R)}{R^2}\bar{\mathcal{W}}_{I_1} \end{pmatrix} . \tag{179}$$

Thus, recalling that $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U} : \boldsymbol{\sigma} = U^a{}_{|bc}\sigma^{bc}\partial_a$, one finds from (178) and (179) that $\nabla^{\boldsymbol{g}}\nabla^{\boldsymbol{g}}\boldsymbol{U} : \boldsymbol{\sigma} = \mathbf{0}$. Also, following (174), the elasticity tensors simplify to

$$\mathbb{C} : \boldsymbol{\epsilon} = 4\bar{\mathcal{W}}_{I_1 I_1}(\boldsymbol{b}^\sharp : \boldsymbol{\epsilon})\boldsymbol{b}^\sharp , \quad \mathbb{D} : \delta\boldsymbol{G} = -2\bar{\mathcal{W}}_{I_1 I_1}(\boldsymbol{C}^\sharp : \delta\boldsymbol{G})\boldsymbol{b}^\sharp - 2\bar{\mathcal{W}}_{I_1}\varphi_{t*}\delta\boldsymbol{G}^\sharp . \tag{180}$$

However, using the incompressibility condition (176), we have $\boldsymbol{b}^\sharp : \boldsymbol{\epsilon} = \frac{1}{R}\left(R\delta r\right)_{,R} + \delta\theta_{,\Theta} - \frac{f}{R^2}\delta z_{,\Theta} = -\frac{f}{R^2}\delta z_{,\Theta}$. Therefore

$$\mathbb{C} : \boldsymbol{\epsilon} = \frac{1}{R^4} \begin{pmatrix} -4R^2 f\bar{\mathcal{W}}_{I_1 I_1}\delta z_{,\Theta} & 0 & 0 \\ 0 & -4f\bar{\mathcal{W}}_{I_1 I_1}\delta z_{,\Theta} & 4f^2\bar{\mathcal{W}}_{I_1 I_1}\delta z_{,\Theta} \\ 0 & 4f^2\bar{\mathcal{W}}_{I_1 I_1}\delta z_{,\Theta} & -4f\left(R^2+f^2\right)\bar{\mathcal{W}}_{I_1 I_1}\delta z_{,\Theta} \end{pmatrix} .$$

On the other hand $\boldsymbol{C}^\sharp : \delta\boldsymbol{G} = -2\frac{f}{R^2}\delta f$, and one can easily obtain

$$\mathbb{D} : \delta\boldsymbol{G} = \begin{pmatrix} \frac{4}{R^2}\bar{\mathcal{W}}_{I_1 I_1} f\delta f & 0 & 0 \\ 0 & \frac{4}{R^4}\bar{\mathcal{W}}_{I_1 I_1} f\delta f & -2\left(\frac{1}{R^2}\bar{\mathcal{W}}_{I_1} + \frac{2f^2}{R^4}\bar{\mathcal{W}}_{I_1 I_1}\right)\delta f \\ 0 & -2\left(\frac{1}{R^2}\bar{\mathcal{W}}_{I_1} + \frac{2f^2}{R^4}\bar{\mathcal{W}}_{I_1 I_1}\right)\delta f & 4\left(\frac{1}{R^2}\bar{\mathcal{W}}_{I_1} + \frac{R^2+f^2}{R^4}\bar{\mathcal{W}}_{I_1 I_1}\right) f\delta f \end{pmatrix} .$$

Therefore, the equilibrium equations (177) simplify to read

$$\operatorname{div}_{\boldsymbol{g}}\delta\boldsymbol{\sigma} = \mathbf{0} , \tag{181}$$

where

$$\begin{aligned} &\delta\sigma^{rr} = \frac{4f\bar{\mathcal{W}}_{I_1 I_1}}{R^2}\left(\delta f - \delta z_{,\Theta}\right) + 4\bar{\mathcal{W}}_{I_1}\delta r_{,R} - \delta p , \quad \delta\sigma^{r\theta} = 2\bar{\mathcal{W}}_{I_1}\left(\frac{\delta r_{,\Theta}}{R^2} + \delta\theta_{,R}\right) , \\ &\delta\sigma^{\theta\theta} = \frac{4f\bar{\mathcal{W}}_{I_1 I_1}}{R^4}\left(\delta f - \delta z_{,\Theta}\right) - \frac{1}{R^2}\left(\delta p + 4\bar{\mathcal{W}}_{I_1}\delta r_{,R}\right) , \\ &\delta\sigma^{\theta z} = -\frac{2\left(2f^2\bar{\mathcal{W}}_{I_1 I_1} + R^2\bar{\mathcal{W}}_{I_1}\right)}{R^4}\left(\delta f - \delta z_{,\Theta}\right) , \quad \delta\sigma^{rz} = 2\bar{\mathcal{W}}_{I_1}\delta z_{,R} , \\ &\delta\sigma^{zz} = \frac{4f}{R^2}[(\bar{\mathcal{W}}_{I_1} + \bar{\mathcal{W}}_{I_1 I_1})\delta f - \bar{\mathcal{W}}_{I_1 I_1}\delta z_{,\Theta}] + \frac{4f^3\bar{\mathcal{W}}_{I_1 I_1}}{R^4}\left(\delta f - \delta z_{,\Theta}\right) - \delta p . \end{aligned} \tag{182}$$

Writing (181) in components along with the incompressibility condition (176) gives the following system of partial differential equations

$$\begin{aligned}&\frac{\partial}{\partial R}\left[\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta z_{,\Theta}-4\bar{\mathcal{W}}_{I_1}\delta r_{,R}+\delta p\right]\\&-\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\left[4R\delta r_{,R}+\delta r_{,\Theta\Theta}+R^2\delta\theta_{,R\Theta}\right]=\frac{\partial}{\partial R}\left[\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta f\right],\end{aligned} \tag{183a}$$

$$\begin{aligned}&\frac{\partial}{\partial R}\left[\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\left(\delta r_{,\Theta}+R^2\delta\theta_{,R}\right)\right]-\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^4}\delta z_{,\Theta\Theta}\\&+\frac{2\bar{\mathcal{W}}_{I_1}}{R^3}\left[3\delta r_{,\Theta}-2R\delta r_{,R\Theta}+3R^2\delta\theta_{,R}\right]-\frac{1}{R^2}\delta p_{,\Theta}=-\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^4}\delta f_{,\Theta},\end{aligned} \tag{183b}$$

$$\begin{aligned}&\frac{\partial}{\partial R}\left[2\bar{\mathcal{W}}_{I_1}\delta z_{,R}\right]+\frac{2\bar{\mathcal{W}}_{I_1}}{R}\delta z_{,R}\\&+\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}+2R^2\bar{\mathcal{W}}_{I_1}}{R^4}\delta z_{,\Theta\Theta}=\left[\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}+\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\right]\delta f_{,\Theta},\end{aligned} \tag{183c}$$

$$\delta r+R\delta r_{,R}+R\delta\theta_{,\Theta}=0\,. \tag{183d}$$

The boundary conditions corresponding to zero incremental boundary traction read $\delta\sigma^{rr}(R_o,\Theta)=0$, $\delta\sigma^{r\theta}(R_o,\Theta)=0$, and $\delta\sigma^{rz}(R_o,\Theta)=0$, which following (182) can be written as

$$\left[4\bar{\mathcal{W}}_{I_1}\delta r_{,R}-\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta z_{,\Theta}-\delta p\right]_{(R_o,\Theta)}=-\left[\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\delta f\right]_{(R_o,\Theta)}, \tag{184a}$$

$$\left[\frac{\delta r_{,\Theta}}{R^2}+\delta\theta_{,R}\right]_{(R_o,\Theta)}=0\,, \tag{184b}$$

$$\delta z_{,R}(R_o,\Theta)=0\,. \tag{184c}$$

In order to fix the rigid body motion of the cylinder, we assume that

$$\delta r(0,\Theta)=0\,,\quad \delta\theta(0,\Theta)=0\,,\quad \delta z(0,\Theta)=0\,. \tag{185}$$

Note that the continuity of the traction across any radial plane of constant Θ gives $\delta\sigma^{\theta z}(R,\Theta)=\delta\sigma^{\theta z}(R,\Theta+2\pi)$, $\delta\sigma^{\theta\theta}(R,\Theta)=\delta\sigma^{\theta\theta}(R,\Theta+2\pi)$, and $\delta\sigma^{\theta r}(R,\Theta)=\delta\sigma^{\theta r}(R,\Theta+2\pi)$. Also, in order to preserve the structural integrity of the cylinder, one must have $\delta r(R,\Theta)=\delta r(R,\Theta+2\pi)$, $\delta\theta(R,\Theta)=\delta\theta(R,\Theta+2\pi)$, and

$\delta z\,(R,\Theta)=\delta z\,(R,\Theta+2\pi)$. Thus, it follows that δr, $\delta\theta$, δz, and δp are 2π-periodic functions with respect to Θ.

Note that $\delta z=\delta z(R,\Theta)$ can be obtained from (183c). Given the solution $\delta z=\delta z(R,\Theta)$ for (183c), we observe that the following functions are the unique solution for the system of linear ordinary differential equations (183) satisfying the boundary conditions (184) and (185):

$$\delta r=0\,,\quad \delta\theta=0\,,\quad \delta p=\frac{4f\bar{\mathcal{W}}_{I_1I_1}}{R^2}\left(\delta f-\delta z_{,\Theta}\right)\,. \tag{186}$$

Therefore, following (182) and (186), the variation of the Cauchy stress tensor reads

$$\delta\boldsymbol{\sigma}=\begin{pmatrix} 0 & 0 & 2\bar{\mathcal{W}}_{I_1}\delta z_{,R} \\ 0 & 0 & -\left(\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}+\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}\right)(\delta f-\delta z_{,\Theta}) \\ 2\bar{\mathcal{W}}_{I_1}\delta z_{,R} & -\left(\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}+\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}\right)(\delta f-\delta z_{,\Theta}) & \frac{4f\bar{\mathcal{W}}_{I_1}}{R^2}\delta f+\frac{4f^3\bar{\mathcal{W}}_{I_1I_1}}{R^4}(\delta f-\delta z_{,\Theta}) \end{pmatrix}. \tag{187}$$

Let us first solve (183c) for $\delta z=\delta z(R,\Theta)$ to complete the solution (186). Recalling that δz is 2π-periodic with respect to Θ and assuming that δf is periodic as well, we can represent them by the following Fourier series

$$\delta z=\sum_{k=-\infty}^{\infty}\delta z_k(R)e^{\mathrm{i}k\Theta}\,,\quad \delta f=\sum_{k=-\infty}^{\infty}\delta f_k(R)e^{\mathrm{i}k\Theta}\,, \tag{188}$$

where $\mathrm{i}=\sqrt{-1}$, and for $k\in\mathbb{Z}$, δz_k and δf_k are the complex-valued Fourier coefficients given by

$$\delta z_k(R)=\frac{1}{2\pi}\int_0^{2\pi}\delta z(R,\zeta)e^{-\mathrm{i}k\zeta}d\zeta\,,\quad \delta f_k(R)=\frac{1}{2\pi}\int_0^{2\pi}\delta f(R,\zeta)e^{-\mathrm{i}k\zeta}d\zeta\,. \tag{189}$$

Substituting the Fourier series (188) into the partial differential equation (183c) for $k\in\mathbb{Z}$, we find

$$2\bar{\mathcal{W}}_{I_1}\delta z_k''+\left[2\frac{d\bar{\mathcal{W}}_{I_1}}{dR}+\frac{2\bar{\mathcal{W}}_{I_1}}{R}\right]\delta z_k'-\left[\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}+\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\right]k^2\delta z_k=\left[\frac{4f^2\bar{\mathcal{W}}_{I_1I_1}}{R^4}+\frac{2\bar{\mathcal{W}}_{I_1}}{R^2}\right]\mathrm{i}k\delta f_k\,, \tag{190}$$

where $\delta z_k{}' = \frac{d\delta z_k}{dR}$, and $\delta z_k{}'' = \frac{d^2\delta z_k}{dR^2}$. Note that δf_k can also be written as $\delta f_k = \frac{1}{2\pi}\int_0^R \xi\delta b_k(\xi)d\xi$, where δb_k is the k^{th} Fourier coefficient of δb . The boundary conditions for δz from (184) and (185) transform in terms of its Fourier coefficients to the following relations

$$\delta z_k(0) = 0\,, \quad \delta z_k{}'(R_o) = 0, \quad k \in \mathbb{Z}. \tag{191}$$

Therefore, we have transformed the real partial differential equation (183c) into a set of complex ordinary differential equations (190).

4.3.3 Energy of a perturbed dislocation distribution

We next calculate the change in energy due to a small perturbation of the defect distribution to the first order in the defect perturbation. For a given distribution of screw dislocations, the energy per unit length in a cylinder made of a generalized neo-Hookean solid is written as

$$W = \int_0^{2\pi}\int_0^{R_o} \mathcal{W}(I_1(R,\Theta))RdRd\Theta\,.$$

Therefore, the variation of the energy following an arbitrary perturbation $\delta b = \delta b(R,\Theta)$ is written as

$$\delta W = \int_0^{2\pi}\int_0^{R_o} \frac{d\mathcal{W}(I_{1\epsilon}(R,\Theta))}{d\epsilon}\bigg|_{\epsilon=0} RdRd\Theta = \int_0^{2\pi}\int_0^{R_o} \mathcal{W}_{I_1}(I_1(R,\Theta))\delta I_1(R,\Theta))RdRd\Theta\,.$$

Note that $\delta I_1 = 2\boldsymbol{\epsilon}\!:\!\boldsymbol{b}^\sharp + \delta\boldsymbol{G}\!:\!\boldsymbol{C}^\sharp = \frac{2f(R)}{R^2}\left[\delta f(R,\Theta) - \delta z_{,\Theta}(R,\Theta)\right]$. Therefore[6]

$$\delta W = \int_0^{R_o}\int_0^{2\pi} \frac{2f(R)}{R}\mathcal{W}_{I_1}(I_1(R))\delta f(R,\Theta)d\Theta dR\,. \tag{192}$$

Remark 4.3.1. Note that (192) can be written as

$$\delta W = \int_0^{R_o} \frac{4\pi f(R)}{R}\mathcal{W}_{I_1}(I_1(R))\delta f_0(R)dR\,, \tag{193}$$

[6]Note that since $\delta z = \delta z(R,\Theta)$ is periodic with repect to Θ, one has $\int_0^{2\pi}\delta z_{,\Theta}(R,\Theta)d\Theta = \delta z(R,2\pi) - \delta z(R,0) = 0$.

where $\delta f_0(R) = \frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\Theta)d\Theta$ is the angular mean value of δf . On the other hand, one can write

$$\delta f_0(R) = \frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\Theta)d\Theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{R} \xi\delta b(\xi,\Theta)d\xi d\Theta = \frac{1}{2\pi}\int_0^{R} \xi\delta b_0(\xi)d\xi,$$

where

$$\delta b_0(R) = \frac{1}{2\pi}\int_0^{2\pi} \delta b(R,\Theta)d\Theta. \tag{194}$$

Hence, the energy variation depends only on $\delta b_0(R)$ —the angular mean value of the perturbation $\delta b(R,\Theta)$.

4.3.4 Perturbed dislocations in incompressible neo-Hookean solids

Let us consider an incompressible homogeneous neo-Hookean solid, i.e., $\bar{\mathcal{W}}(I_1) = \frac{\mu}{2}(I_1 - 3)$, where μ is the shear modulus for infinitesimal strains, and an arbitrary perturbation $\delta b = \delta b(R,\Theta)$.

Remark 4.3.2. Note that even though the energy per unit length along a single screw dislocation line in a neo-Hookean solid is unbounded as shown in [98] (see also [87]), energy is not necessarily unbounded for distributed screw dislocations. In particular, for a radially-symmetric distribution of screw dislocations, the energy per unit length in a neo-Hookean solid is written as

$$W = 2\pi\int_0^{R_o} \frac{\mu}{2}\left(I_1(\xi) - 3\right)\xi d\xi = \pi\mu\int_0^{R_o}\frac{f^2(\xi)}{\xi}d\xi\,.$$

Let us assume, as an example for computing the energy, the following Burgers' vector distribution

$$b(R) = \begin{cases} b_i & 0 < R \le R_i\,, \\ 0 & R_i < R \le R_o\,, \end{cases} \tag{195}$$

where $R_i \le R_o$ is the radius of a cylinder made of a solid with a uniform Burgers' vector b_i , while the hollow cylinder $R_i < R \le R_o$ is dislocation-free. Thus, one finds

$$f(\xi) = \frac{1}{2\pi}\int_0^{\xi} \zeta b(\zeta)d\zeta = \begin{cases} \dfrac{b_i\xi^2}{4\pi} & 0 \le \xi \le R_i\,, \\ \dfrac{b_i R_i^2}{4\pi} & R_i < \xi \le R_o\,. \end{cases} \tag{196}$$

Therefore

$$W = \pi\mu \int_0^{R_i} \frac{1}{\xi}\left(\frac{b_i \xi^2}{4\pi}\right)^2 d\xi + \pi\mu \int_{R_i}^{R_o} \frac{1}{\xi}\left(\frac{b_i R_i^2}{4\pi}\right)^2 d\xi = \frac{\mu b_i^2 R_i^4}{64\pi}\left[1 + 4\log\left(\frac{R_o}{R_i}\right)\right] < \infty\,.$$

In the following computation, we consider an arbitrary radially-symmetric Burgers' vector distribution $b = b(R)$ and an arbitrary perturbation $\delta b = \delta b(R,\Theta)$. For a neo-Hookean solid, the ordinary differential equations (190) for $k \in \mathbb{Z}$ simplify and read

$$R^2 \delta z_k{}'' + R\delta z_k{}' - k^2 \delta z_k = \mathrm{i}k\delta f_k\,. \tag{197}$$

Solving (197), one finds that for $k \in \mathbb{Z}$

$$\begin{aligned}\delta z_k(R) = \frac{R^{2k} + R_o^{2k}}{2R^k R_o^k}&\left[c_k + \mathrm{i}\int_{\frac{R}{R_o}}^1 \frac{\left(\xi^k - \xi^{-k}\right)\delta f_k(R_o\xi)}{2\xi} d\xi\right] \\ + \frac{R^{2k} - R_o^{2k}}{2R^k R_o^k}&\left[d_k - \mathrm{i}\int_{\frac{R}{R_o}}^1 \frac{\left(\xi^k + \xi^{-k}\right)\delta f_k(R_o\xi)}{2\xi} d\xi\right],\end{aligned} \tag{198}$$

for some complex constants c_k and d_k. By using the boundary condition (191) $\delta z_k{}'(R_o) = 0$, it follows that $d_k = 0$. We observe that $c_k = \delta z_k(R_o)$, and from (189), one observes that $\delta z_{-k} = \delta z_k^*$.[7] Thus, $c_{-k} = c_k^*$. Also, note that $\delta f_{-k} = \delta f_k^*$. Therefore, following (198) and by using (188), it follows that

$$\begin{aligned}\delta z(R,\Theta) = c_0 + \sum_{k=1}^{\infty} \frac{R^{2k} + R_o^{2k}}{2R^k R_o^k}&\Bigg[2\Big(\Re(c_k)\cos(k\Theta) - \Im(c_k)\sin(k\Theta)\Big) \\ &- \int_{\frac{R}{R_o}}^1 \frac{\left(\xi^k - \xi^{-k}\right)\Big(\Re[\delta f_k(R_o\xi)]\sin(k\Theta) + \Im[\delta f_k(R_o\xi)]\cos(k\Theta)\Big)}{\xi} d\xi\Bigg] \\ + \sum_{k=1}^{\infty} \frac{R^{2k} - R_o^{2k}}{2R^k R_o^k} \int_{\frac{R}{R_o}}^1 &\frac{\left(\xi^k + \xi^{-k}\right)\Big(\Re[\delta f_k(R_o\xi)]\sin(k\Theta) + \Im[\delta f_k(R_o\xi)]\cos(k\Theta)\Big)}{\xi} d\xi\,,\end{aligned} \tag{199}$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number z, respectively. Note that since $\delta f_k(R) = \frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\zeta) e^{-\mathrm{i}k\zeta} d\zeta$, one can write

$$\Re\left[\delta f_k(R)\right] = \frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\Theta)\cos(k\Theta)\, d\Theta\,, \quad \Im\left[\delta f_k(R)\right] = -\frac{1}{2\pi}\int_0^{2\pi} \delta f(R,\Theta)\sin(k\Theta)\, d\Theta\,.$$

[7] We denote by x^* the complex conjugate of a complex number x.

Remark 4.3.3. Note that for a neo-Hookean solid, the incremental deformation is independent of the finite radially-symmetric dislocation distribution $b = b(R)$. Indeed, the governing equation (197) holds for any $b = b(R)$. However, as can be seen in (187), the incremental stress field, and in particular $\delta\sigma^{zz}$, depends on the initial dislocation distribution.

Let us now simplify the solution (199) for a particular Burgers' vector perturbation given by

$$\delta b(R,\Theta) = \delta b_0(R) + \frac{R}{R_o}\left(1-\frac{R}{R_o}\right)^2 \left[b_1\cos\Theta + b_2\sin\Theta\right], \tag{200}$$

for some R-dependent function $\delta b_0 = \delta b_0(R)$, and constants b_1 and b_2. Note that the only non-zero Fourier coefficients of δb in (200) are δb_0, δb_1, and δb_{-1}. For $k = -1, 1$, one finds

$$\delta b_k(R) = \frac{1}{2}\left(b_1 - \mathrm{i}kb_2\right)\frac{R}{R_o}\left(1-\frac{R}{R_o}\right)^2 .$$

Therefore, the only non-zero Fourier coefficients of δf are δf_0, δf_1, and δf_{-1}. They read

$$\delta f_0(R) = \frac{1}{2\pi}\int_0^R \xi\delta b_0(\xi)d\xi \quad \text{and} \quad \delta f_k(R) = \frac{b_1 - \mathrm{i}kb_2}{4\pi}\left(\frac{R^5}{5R_o^3} - \frac{R^4}{2R_o^2} + \frac{R^3}{3R_o}\right) \quad \text{for} \quad k = -1, 1 . \tag{201}$$

First, note that following (198), for $k \neq -1, 1$ one obtains $\delta z_k(R) = c_k\frac{R^{2k}+R_o^{2k}}{2R^kR_o^k}$. However, since we are looking for a solution that is bounded, it follows that for $k \neq -1, 0, 1$, one has $c_k = 0$. Thus, one finds following (199) that

$$\begin{aligned}\delta z(R,\Theta) = c_0 &- \frac{b_1\left(R-R_o\right)^2\left(R^4 - 2R_oR^3 + 2R_o^3R + R_o^4\right) + 240\pi\Im(c_1)R_o^2\left(R^2+R_o^2\right)}{240\pi R_o^3 R}\sin\Theta \\ &+ \frac{b_2\left(R-R_o\right)^2\left(R^4 - 2R_oR^3 + 2R_o^3R + R_o^4\right) + 240\pi\Re(c_1)R_o^2\left(R^2+R_o^2\right)}{240\pi R_o^3 R}\cos\Theta .\end{aligned} \tag{202}$$

Further, to ensure that (202) is bounded, one must have $b_1R_o^2 + 240\pi\Im(c_1) = 0$, and $b_2R_o^2 + 240\pi\Re(c_1) = 0$. Thus, $\Im(c_1) = -b_1R_o^2/(240\pi)$, and $\Re(c_1) = -b_2R_o^2/(240\pi)$.

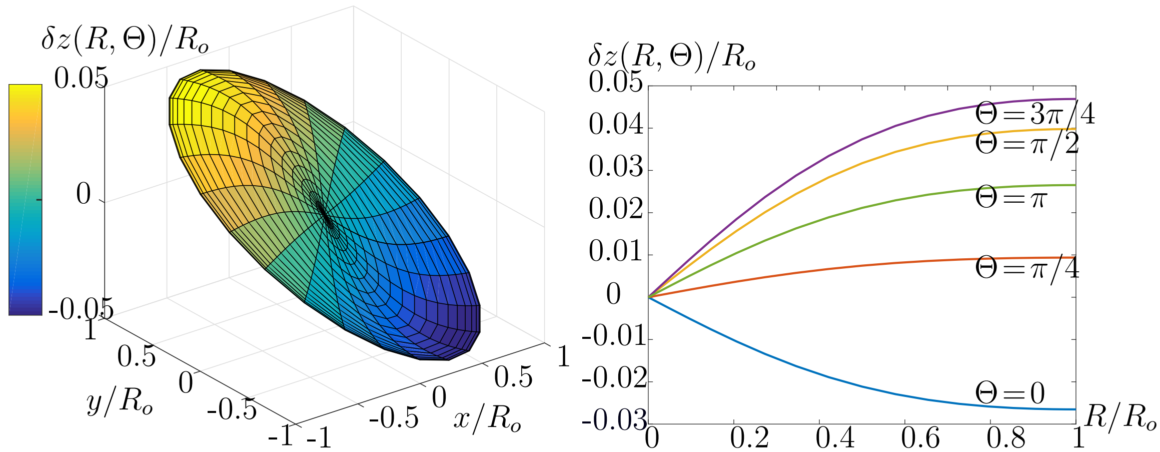


Figure 20: Visualization of the solution (203) for a cylinder of radius R_o with $b_1R_o = 15$ and $b_2R_o = 10$. Left: 3D visualization of the deformation of a cross section of the cylinder. Right: Profile of deformation of different radial lines.

Next, by enforcing the boundary condition (191) $\delta z(0,\Theta) = 0$ to fix the rigid body motion, one finds $c_0 = 0$. Therefore, it follows that

$$\delta z(R,\Theta) = \frac{b_2\cos\Theta - b_1\sin\Theta}{240\pi R_o^3} R\left(R^4 - 4R^3R_o + 5R^2R_o^2 - 4R_o^4\right) . \tag{203}$$

In Fig. 20, we plot the solution (203) for a cylinder of radius R_o subject to a perturbation (200) such that $b_1R_o = 15$ and $b_2R_o = 10$. Note that the numerical values shown in Fig. 20 should be multiplied by a small ϵ to give the incremental deformation. Given that $z = Z$ for the finite dislocation distribution, the total deformation reads: $z_\epsilon = Z + \epsilon\delta z + o(\epsilon)$. Recall, as noted earlier, that the state of deformation of a cylinder made of a neo-Hookean solid is independent of $b = b(R)$; it only depends on the perturbation —compare this to Example **??** where the deformation of a cylinder made of a power law material actually depends on the finite dislocation distribution $b = b(R)$.

Using (175) and (187), one finds the following total stress in the perturbed configuration (recall that the total stress in the perturbed configuration for a small enough

ϵ is $\boldsymbol{\sigma}_\epsilon = \boldsymbol{\sigma} + \epsilon\delta\boldsymbol{\sigma} + o(\epsilon)$.)

$$\boldsymbol{\sigma}_\epsilon = \begin{pmatrix} 0 & 0 & \epsilon\mu\delta z_{,R} \\ 0 & 0 & -\mu\frac{f(R)}{R^2} - \epsilon\frac{\mu}{R^2}\left(\delta f - \delta z_{,\Theta}\right) \\ \epsilon\mu\delta z_{,R} & -\mu\frac{f(R)}{R^2} - \epsilon\frac{\mu}{R^2}\left(\delta f - \delta z_{,\Theta}\right) & \mu\frac{f^2(R)}{R^2} + \epsilon\frac{2\mu}{R^2} f\delta f \end{pmatrix} + o(\epsilon)\,,$$

where

$$\begin{aligned} \delta f &= \frac{1}{2\pi}\int_0^R \xi\delta b_0(\xi)d\xi + \left(\frac{R^5}{5R_o^3} - \frac{R^4}{2R_o^2} + \frac{R^3}{3R_o}\right)\frac{b_1\cos\Theta + b_2\sin\Theta}{2\pi}\,, \\ \delta z_{,R} &= \left(5R^4 - 16R^3R_o + 15R^2R_o^2 - 4R_o^4\right)\frac{b_1\sin\Theta - b_2\cos\Theta}{240\pi R_o^3}\,, \\ \delta f - \delta z_{,\Theta} &= \frac{1}{2\pi}\int_0^R \xi\delta b_0(\xi)d\xi + R\left(23R^4 - 56R^3R_o + 35R^2R_o^2 + 4R_o^4\right)\frac{b_1\cos\Theta + b_2\sin\Theta}{240\pi R_o^3}\,. \end{aligned}$$

Let us now compute the variation of the energy due to a dislocation distribution perturbation. Following (193), one has

$$\delta W = \int_0^{R_o} \frac{2\pi\mu}{R} f(R)\delta f_0(R)dR\,.$$

Assuming the finite dislocation distribution (195), the variation of the energy reads

$$\delta W = \int_0^{R_i} \frac{\mu b_i R}{4\pi}\int_0^R \xi\delta b_0(\xi)d\xi dR + \int_{R_i}^{R_o} \frac{\mu b_i R_i^2}{4R\pi}\int_0^R \xi\delta b_0(\xi)d\xi dR\,.$$

Let us assume that the total Burgers' vector of the perturbation is zero so that the perturbation does not change the total Burgers' vector of the original finite dislocation distribution $b(R)$, i.e., $\int_0^{R_o}\int_0^{2\pi} R\delta b(R,\Theta)d\Theta dR = 0$. In terms of the angular mean value of the perturbation this is written as $\int_0^{R_o} R\delta b_0(R)dR = 0$. We consider in particular a Burgers' vector perturbation such that its angular mean value —cf. (194) —is given by

$$\delta b_0(R) = 15 b_0 \frac{R}{R_o}\left(1 - \frac{R}{R_o}\right)^2\left(1 - 2\frac{R}{R_o}\right)\,, \tag{204}$$

for some constant b_0. For this perturbation one obtains

$$\delta W = \frac{\mu b_i b_o\left(35R_i^8 - 144R_i^7R_o + 210R_i^6R_o^2 - 112R_i^5R_o^3 + 14R_i^2R_o^6\right)}{672\pi R_o^4}\,.$$

Note that for any R_i such that $0 < R_i < R_o$, the energy variation δW has the same sign as $b_i b_o$. For $R_i = 0$, $\delta W = 0$ and $\delta W/(b_i b_o)$ is monotonically increasing as a function of R_i. In particular, for $R_i > 0$, $\delta W \neq 0$, and hence the initial dislocation distribution is not in equilibrium.

CHAPTER V

MOTION OF AN ELASTIC BODY IN A DEFORMING AMBIENT SPACE

In this section we formulate a nonlinear elasticity theory in which the ambient space is deforming. For a continuum moving in an evolving ambient space, we model time dependency of the metric by a time-dependent embedding of the ambient space in a larger manifold with a fixed background metric. We derive both the tangential and the normal governing equations. We then reduce the standard energy balance written in the larger ambient space to that in the evolving ambient space. We consider quasi-static deformations of the ambient space and find closed-form solution for the case of a spherical cap sitting on a sphere with quasi-statically evolving radius. Note that the results of this section have been previously reported in our published work [97].

5.1 Lagrangian Field Theory of Elasticity in an Evolving Ambient Space

We identify the reference configuration of an elastic body with a Riemannian manifold $(\mathcal{B}, \boldsymbol{G})$ and let the body deform in a time-dependent ambient space $\mathcal{S}_t$, which is evolving in a Euclidean space $(\mathcal{Q}, \boldsymbol{h})$ of higher dimension. The evolution of the ambient space $\mathcal{S}_t$ is given by a time-dependent embedding $\psi_t : \mathcal{S} \to \mathcal{Q}$, for some abstract manifold $\mathcal{S}$, such that $\psi_t(\mathcal{S}) = \mathcal{S}_t$, and the evolving metric of the ambient space $\mathcal{S}$ is given as the induced metric by that of $\mathcal{Q}$, i.e., $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$, which means that ψ_t is an isometric embedding.[1] See Fig. 21. We denote inner products of vectors with respect to the metrics $\boldsymbol{h}$ and $\boldsymbol{g}_t$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{h}}$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{g}_t}$, respectively. We denote

[1]Note that for a given t, such an isometric embedding always exists for $\dim \mathcal{Q}$ large enough, by Nash's embedding theorem [54].

the local coordinates on $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{Q}$ by $\{X^A\}$, $\{x^a\}$, and $\{\chi^\alpha\}$, respectively. Let $\dim \mathcal{S}_t = n$, and $\dim \mathcal{Q} = n + k = m$. Let $\{\boldsymbol{\eta}_i^t\}_{i=1,...,k}$ be a smooth orthonormal basis for $\mathfrak{X}^\perp(\mathcal{S}_t)$, the set of vector fields normal to $\mathcal{S}_t$. Let $\{\chi^\alpha\}$ be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1, ..., \chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$ and such that the unit normal vector field $\boldsymbol{\eta}_i^t$, for $i \in \{1, ..., k\}$ is tangent to the coordinate curve χ^{n+i}, for $i \in \{1, ..., k\}$. Recall, as discussed in Appendix A.3, that every vector field $\boldsymbol{u}$ on $\mathcal{Q}$ along $\mathcal{S}_t$ can be written as $\boldsymbol{u} = \boldsymbol{u}_\parallel + \sum_{i=1}^k u_\perp^i \boldsymbol{\eta}_i^t$, where $\boldsymbol{u}_\parallel$ is the part of the vector $\boldsymbol{u}$ tangential to $\mathcal{S}_t$, and $u_\perp^i = u^{n+i}$ for $i \in \{1, ..., k\}$. Also, recall that for $i, j \in \{1, ..., k\}$, and $\alpha \in \{1, ..., n + k\}$, we have $\langle\!\langle \boldsymbol{\eta}_i^t, \boldsymbol{\eta}_j^t \rangle\!\rangle_{\boldsymbol{h}} = \delta_{ij}$, $\langle\!\langle \boldsymbol{\eta}_i^t, \boldsymbol{u}_\parallel \rangle\!\rangle_{\boldsymbol{h}} = 0$, and $h_{\alpha(n+i)} = \delta_{\alpha(n+i)}$. For $i \in \{1, ..., k\}$, we denote the i^{th} second fundamental form of $\mathcal{S}_t$ along the unit normal $\boldsymbol{\eta}_i^t$ by $\boldsymbol{\kappa}_i^t$ and let $\boldsymbol{k}_i^t = \psi_t^* \boldsymbol{\kappa}_i^t$. For $i, j \in \{1, ..., k\}$, we denote the normal fundamental 1-form of $\mathcal{S}_t$ relative to the unit normals $\boldsymbol{\eta}_i^t$ and $\boldsymbol{\eta}_j^t$ in this order by $\boldsymbol{\omega}_{ij}^t$ and let $\boldsymbol{o}_{ij}^t = \psi_t^* \boldsymbol{\omega}_{ij}^t$.[2] We define a motion of $(\mathcal{B}, \boldsymbol{G})$ in $\left(\mathcal{S}_t, \boldsymbol{h}|_{\mathcal{S}_t}\right)$ as a one-parameter family of maps $\tilde{\varphi}_t : \mathcal{B} \to \mathcal{S}_t$, where t is time. This is equivalent to a motion of $(\mathcal{B}, \boldsymbol{G})$ in $(\mathcal{S}, \boldsymbol{g}_t)$: $\varphi_t : \mathcal{B} \to \mathcal{S}$, such that $\varphi_t = \psi_t^{-1} \circ \tilde{\varphi}_t$. We let $\tilde{\varphi}(X, t) := \tilde{\varphi}_t(X)$, $\varphi(X, t) := \varphi_t(X)$, and $\psi(x, t) := \psi_t(x)$. Let $\{\tilde{\partial}_\alpha^t\}_{\alpha=1,...,n}$ and $\{\partial_a^t\}_{a=1,...,n}$ denote local coordinate bases for $\mathcal{S}_t$ and $\mathcal{S}$, respectively.

In order to describe the dynamics of the motion of $\mathcal{B}$, the Lagrangian field theory should be formulated with respect to the fixed space $\mathcal{Q}$. For an elastic material, the Lagrangian density $\mathcal{L}$ can be written as[3]

$$\mathcal{L} = \mathcal{L}(\boldsymbol{X}, \tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}),$$

where $\tilde{\boldsymbol{F}} = T\tilde{\varphi}_t = \psi_{t*}\boldsymbol{F}$ and $\boldsymbol{F} = T\varphi_t$ are the deformation gradients of $\tilde{\varphi}_t$ and φ_t,

[2]Recall that the order matters since $\boldsymbol{\omega}_{ij}^t = -\boldsymbol{\omega}_{ji}^t$. See Appendix A.3 for more details and the definitions of both the second and the normal fundamental forms.

[3]Note that although the Lagrangian theory is formulated with respect to $\mathcal{Q}$, the density is defined with respect to the volume element of $\mathcal{B}$, i.e., $\mathcal{L}$, is an n-dimensional density, not an m-dimensional one.

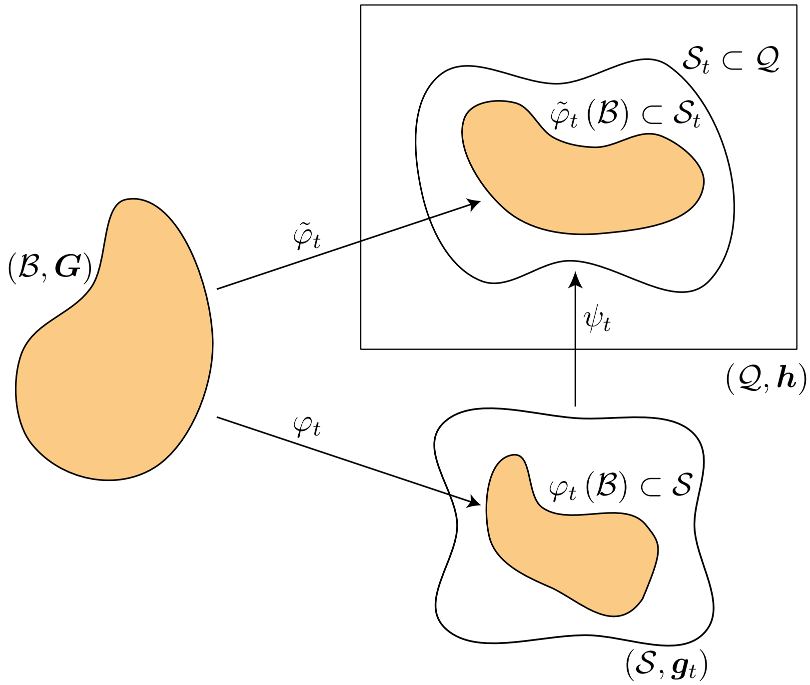


Figure 21: Motion of an elastic body in an evolving ambient space.

respectively. We assume that the Lagrangian density can be written as

$$\mathcal{L} = \frac{1}{2}\rho_0 \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} - \rho_0 W(\boldsymbol{X}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}), \tag{205}$$

where ρ_0 is the material mass density, $\boldsymbol{\Upsilon} := \dot{\tilde{\varphi}} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta} \circ \varphi_t$ is the material velocity vector field of $\tilde{\varphi}$, $\boldsymbol{V}$ is the material velocity vector field of φ, $\boldsymbol{\zeta} = \partial\psi/\partial t$ is the velocity of a given, fixed point $x \in \mathcal{S}$ as it moves in $\mathcal{Q}$, and $W = W(\boldsymbol{X}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h})$ is the elastic energy density (energy function).

Remark 5.1.1. Note that since $\boldsymbol{g}_t := \psi_t^*\boldsymbol{h}$, i.e., ψ_t is an isometry between $(\mathcal{S}, \boldsymbol{g}_t)$ and $(\mathcal{S}_t, \boldsymbol{h})$, by objectivity (the isometry ψ_t can be interpreted as a change of observer), the dependence of the elastic energy on $\tilde{\boldsymbol{F}} = \psi_{t*}\boldsymbol{F}$ reduces to a dependence on $\boldsymbol{F}$ only. It should also depend on $\boldsymbol{G}$ and $\boldsymbol{g}_t$ (instead of $\boldsymbol{h}$) so that one can get a scalar out of $\boldsymbol{F}$. Hence, we have[4]

$$W(\boldsymbol{X}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}) = W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t)\,. \tag{206}$$

For a continuum under a body force field $\boldsymbol{\beta}$ (not necessarily conservative), the Lagrange-d'Alembert principle states that [49]

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} dV dt + \int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 \boldsymbol{\beta}^\flat \cdot \delta\tilde{\varphi} dV dt = 0\,, \tag{207}$$

where $^\flat$ denotes the flat operator for lowering tensor indices, $\boldsymbol{\beta}$ denotes body force per unit mass, and dV denotes the volume element for $\mathcal{B}$. Note that $\boldsymbol{\beta}$ is not necessarily tangent to $\mathcal{S}_t$ and we write it as $\boldsymbol{\beta} = \boldsymbol{\beta}_\parallel + \sum_{i=1}^{k} B_\perp^i \boldsymbol{\eta}_i^t$, where $\boldsymbol{\beta}_\parallel$ is the part of $\boldsymbol{\beta}$ tangent to $\mathcal{S}_t$, and $B_\perp^i$, for $i \in \{1, ..., k\}$, is its component along the i^{th} normal $\boldsymbol{\eta}_i^t$. The action is defined on the material manifold $(\mathcal{B}, \boldsymbol{G})$ as

$$S\left(\tilde{\varphi}\right) = \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L}(\boldsymbol{X}, \tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}) dV(\boldsymbol{X}) dt\,, \tag{208}$$

where $dV(\boldsymbol{X})$ is the Riemannian volume element on $\mathcal{B}$. For the assumed Lagrangian (205), we have $S = S_T + S_W$, where

$$\begin{aligned} S_T &= \int_{t_0}^{t_1} \int_{\mathcal{B}} \frac{1}{2} \rho_0 \left\langle\!\left\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} dV(\boldsymbol{X}) dt\,, \\ S_W &= -\int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t) dV(\boldsymbol{X}) dt\,. \end{aligned}$$

In order to take variations of the action (208), we consider a variation field $\tilde{\varphi}_\epsilon$ of $\tilde{\varphi}$ such that $\tilde{\varphi}_0 = \tilde{\varphi}$ and define its variation as

$$\delta\tilde{\varphi} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \tilde{\varphi}_\epsilon\,.$$

First, we look at the resulting variations of the kinetic energy

$$\frac{d}{d\epsilon} \frac{1}{2} \left\langle\!\left\langle \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \right\rangle\!\right\rangle_{\boldsymbol{h}} = \left\langle\!\left\langle D_\epsilon^{\boldsymbol{h}} \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \right\rangle\!\right\rangle_{\boldsymbol{h}}\,,$$

[4]Another way to see this is by looking at the elastic energy as a function of the right Cauchy-Green tensor, i.e., $W = \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}, \boldsymbol{G})$. First, we see that since $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$, then $\left(\psi_t \circ \varphi_t\right)^* \boldsymbol{h} = \varphi_t^* \psi_t^* \boldsymbol{h} = \varphi_t^* \boldsymbol{g}_t$, i.e., the right Cauchy-Green tensors $\boldsymbol{C}$ of φ_t and $\tilde{\boldsymbol{C}}$ of $\tilde{\varphi}_t$ are equal. If we denote $\boldsymbol{f} := T\psi_t$, we write in components $\tilde{C}_{AB} = f^\alpha{}_a F^a{}_A f^\beta{}_b F^b{}_B h_{\alpha\beta} = F^a{}_A F^b{}_B f^\alpha{}_a f^\beta{}_b h_{\alpha\beta} = F^a{}_A F^b{}_B g_{ab} = C_{AB}$. Therefore, $W = \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}, \boldsymbol{G}) = \tilde{W}(\boldsymbol{X}, \boldsymbol{C}, \boldsymbol{G})$, that is, the elastic energy does not depend on the embedding ψ_t.

where $D_\epsilon^{\boldsymbol{h}}$ denotes the covariant derivative along the curve $\epsilon \mapsto \tilde{\varphi}_\epsilon(X,t)$ for fixed X and t. Using the symmetry lemma, we have $D_\epsilon^{\boldsymbol{h}}\boldsymbol{\zeta}_\epsilon = D_t^{\boldsymbol{h}}\delta\tilde{\varphi}$, where $D_t^{\boldsymbol{h}}$ denotes the covariant derivative along the curve $t \mapsto \tilde{\varphi}(X,t)$ for fixed X. Therefore, we can write

$$\frac{d}{d\epsilon}\frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \rangle\!\rangle_{\boldsymbol{h}} = \left\langle\!\!\left\langle D_t^{\boldsymbol{h}}\delta\tilde{\varphi}, \boldsymbol{\Upsilon} \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \frac{d}{dt}\langle\!\langle \delta\tilde{\varphi}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} - \left\langle\!\!\left\langle \delta\tilde{\varphi}, D_t^{\boldsymbol{h}}\boldsymbol{\Upsilon} \right\rangle\!\!\right\rangle_{\boldsymbol{h}}.$$

Assuming that the variation of $\tilde{\varphi}$ is fixed at t_0 and t_1, i.e., $\delta\tilde{\varphi}(t_0) = \delta\tilde{\varphi}(t_1) = 0$, the first term on the right-hand side does not contribute to the action. We decompose the velocity $\boldsymbol{\Upsilon}$ into tangent and normal components as $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_\parallel + \boldsymbol{\Upsilon}_\perp$, where $\boldsymbol{\Upsilon}_\parallel = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta}_\parallel \circ \varphi_t$ and $\boldsymbol{\Upsilon}_\perp = \sum_{i=1}^k \zeta_\perp^i \boldsymbol{\eta}_i^t$, such that $\boldsymbol{\zeta}$ is written in terms of its tangent and normal components as $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\parallel + \sum_{i=1}^k \zeta_\perp^i \boldsymbol{\eta}_i^t$. We denote the acceleration in $\mathcal{Q}$ by $\boldsymbol{\Gamma} = D_t^{\boldsymbol{h}}\boldsymbol{\Upsilon}$ and decompose it into tangent and normal components with respect to $\mathcal{S}_t$ as $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_\parallel + \sum_{i=1}^k \Gamma_\perp^i \boldsymbol{\eta}_i^t$. We denote by $\boldsymbol{A} = \psi_t^*\boldsymbol{\Gamma}_\parallel$ the intrinsic acceleration of $\mathcal{S}$. Therefore, the variation of the kinetic energy is calculated as

$$\begin{aligned}\delta\left(\frac{1}{2}\rho_0\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}}\right) &= \frac{d}{dt}\langle\!\langle \delta\tilde{\varphi}, \rho_0\boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} - \left\langle\!\!\left\langle \delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{\Gamma}_\parallel \right\rangle\!\!\right\rangle_{\boldsymbol{h}} - \rho_0\sum_{i=n+1}^m \Gamma_\perp^i\delta\tilde{\varphi}_\perp^i \\ &= \frac{d}{dt}\langle\!\langle \delta\tilde{\varphi}, \rho_0\boldsymbol{\Upsilon} \rangle\!\rangle_{\boldsymbol{h}} - \left\langle\!\!\left\langle \psi^*\delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{A} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} - \rho_0\sum_{i=n+1}^m \Gamma_\perp^i\delta\tilde{\varphi}_\perp^i,\end{aligned}$$

where $\delta\tilde{\varphi}_\parallel$ is the part of $\delta\tilde{\varphi}$ tangent to $\mathcal{S}_t$ and $\delta\tilde{\varphi}_\perp^i$ is its component along $\boldsymbol{\eta}_i^t$, for $i \in \{1, ..., k\}$. Assuming that the variation of $\tilde{\varphi}$ is fixed on the boundary, i.e., $\delta\tilde{\varphi}|_{\partial\varphi(\mathcal{B})} = 0$, we obtain

$$\delta S_T = -\int_{t_0}^{t_1}\int_{\mathcal{B}}\left(\left\langle\!\!\left\langle \psi^*\delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{A} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} + \rho_0\sum_{i,j=n+1}^m \Gamma_\perp^i\delta\tilde{\varphi}^j\delta_{ij}\right)dV(\boldsymbol{X})dt. \tag{209}$$

Next we compute the components of the acceleration.

Proposition 5.1.1. *The tangent and normal accelerations are given by*

$$\begin{aligned}\boldsymbol{A} &= D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} + 2\sum_{i=1}^{k}\zeta^i_\perp \boldsymbol{g}_t^\sharp\cdot\boldsymbol{k}^t_i\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right)\\ &\quad -\sum_{i=1}^{k}\zeta^i_\perp\left(d\zeta^i_\perp\right)^\sharp - \sum_{i,j=1}^{k}\zeta^i_\perp\zeta^j_\perp\boldsymbol{o}^{t\sharp}_{ij}\\ &= D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}\right]^{\mathsf{T}}\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \boldsymbol{g}_t^\sharp\cdot\frac{\partial\boldsymbol{g}_t}{\partial t}\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right)\\ &\quad -\sum_{i=1}^{k}\zeta^i_\perp\left(d\zeta^i_\perp\right)^\sharp - \sum_{i,j=1}^{k}\zeta^i_\perp\zeta^j_\perp\boldsymbol{o}^{t\sharp}_{ij}\,,\end{aligned} \tag{210a}$$

$$\begin{aligned}\Gamma^i_\perp &= \frac{d\zeta^i_\perp}{dt} + d\zeta^i_\perp\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \boldsymbol{k}^t_i\left(\boldsymbol{V}+\boldsymbol{Z},\boldsymbol{V}+\boldsymbol{Z}\right) - 2\sum_{j=1}^{k}\zeta^j_\perp\boldsymbol{o}^t_{ij}\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right)\\ &\quad +\sum_{j,l=1}^{k}\zeta^j_\perp\zeta^l_\perp\left\langle\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j}\boldsymbol{\eta}^t{}_l,\boldsymbol{\eta}^t_i\right\rangle\!\right\rangle_{\boldsymbol{h}}\,,\end{aligned} \tag{210b}$$

where $i=1,...,k$, d denotes the exterior derivative on $\mathcal{S}$, i.e., $d\zeta^i_\perp = \sum_{a=1}^{n}\frac{\partial\zeta^i_\perp}{\partial x^a}dx^a$, $D_t^{\boldsymbol{g}_t}$ denotes the covariant derivative along the curve $t\mapsto\varphi\left(X,t\right)$ for fixed X , $\boldsymbol{Z} := \left(\psi_t^*\boldsymbol{\zeta}_\parallel\right)\circ\varphi_t$ is the tangent part of the velocity $\boldsymbol{\zeta}$, and $^{\mathsf{T}}$ denotes the transpose operator with respect to the metric $\boldsymbol{g}_t$.

Remark 5.1.2. Before we proceed to the proof, let us first look at some particular cases. If we assume that the evolution of the ambient space is transversal, i.e., $\boldsymbol{Z}=\boldsymbol{0}$, then (215) reduces to

$$\begin{aligned}\boldsymbol{A} &= D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\sum_{i=1}^{k}\zeta^i_\perp\boldsymbol{g}_t^\sharp\cdot\boldsymbol{k}^t_i\cdot\boldsymbol{V} - \sum_{i=1}^{k}\zeta^i_\perp\left(d\zeta^i_\perp\right)^\sharp - \sum_{i,j=1}^{k}\zeta^i_\perp\zeta^j_\perp\boldsymbol{o}^{t\sharp}_{ij}\\ &= D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \boldsymbol{g}_t^\sharp\cdot\frac{\partial\boldsymbol{g}_t}{\partial t}\cdot\boldsymbol{V} - \sum_{i=1}^{k}\zeta^i_\perp\left(d\zeta^i_\perp\right)^\sharp - \sum_{i,j=1}^{k}\zeta^i_\perp\zeta^j_\perp\boldsymbol{o}^{t\sharp}_{ij}\,,\end{aligned} \tag{211a}$$

$$\Gamma^i_\perp = \frac{d\zeta^i_\perp}{dt} + d\zeta^i_\perp\cdot\boldsymbol{V} - \boldsymbol{k}^t_i\left(\boldsymbol{V},\boldsymbol{V}\right) - 2\sum_{j=1}^{k}\zeta^j_\perp\boldsymbol{o}^t_{ij}\cdot\boldsymbol{V} + \sum_{j,l=1}^{k}\zeta^j_\perp\zeta^l_\perp\left\langle\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j}\boldsymbol{\eta}^t{}_l,\boldsymbol{\eta}^t_i\right\rangle\!\right\rangle_{\boldsymbol{h}}\,, \tag{211b}$$

where $i=1,...,k$. If we assume that $\mathcal{S}_t$ is a hyperspace in $\mathcal{Q}$, i.e., the co-dimension is $k=1$, the normal fundamental 1-forms reduce to a vanishing 1-form $\boldsymbol{o}^t_{11}=\boldsymbol{0}$, and $\sum_{j,l=1}^{k} \zeta_\perp^j \zeta_\perp^l \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_j^t} \boldsymbol{\eta}^t{}_l, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \zeta_\perp \zeta_\perp \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t} \boldsymbol{\eta}^t, \boldsymbol{\eta}^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \mathbf{0}$, since $\langle\!\langle \boldsymbol{\eta}^t, \boldsymbol{\eta}^t \rangle\!\rangle_{\boldsymbol{h}} = 1$. Therefore, (215) reduces to

$$\begin{aligned} \boldsymbol{A} &= D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} + 2\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot \left(\boldsymbol{V}+\boldsymbol{Z}\right) - \zeta_\perp \left(d\zeta_\perp\right)^\sharp \\ &= D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}\right]^{\mathsf{T}} \cdot \left(\boldsymbol{V}+\boldsymbol{Z}\right) + \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \left(\boldsymbol{V}+\boldsymbol{Z}\right) - \zeta_\perp \left(d\zeta_\perp\right)^\sharp , \end{aligned} \tag{212a}$$

$$\Gamma_\perp = \frac{d\zeta_\perp}{dt} + d\zeta_\perp \cdot \left(\boldsymbol{V}+\boldsymbol{Z}\right) - \boldsymbol{k}^t\left(\boldsymbol{V}+\boldsymbol{Z}, \boldsymbol{V}+\boldsymbol{Z}\right) . \tag{212b}$$

Finally, if we assume that $\mathcal{S}_t$ is a hyperspace evolving transversally in $\mathcal{Q}$, i.e., $k=1$ and $\boldsymbol{Z}=\mathbf{0}$, then, (212) reduce to

$$\begin{aligned} \boldsymbol{A} &= D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot \boldsymbol{V} - \zeta_\perp \left(d\zeta_\perp\right)^\sharp \\ &= D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} - \zeta_\perp \left(d\zeta_\perp\right)^\sharp , \end{aligned} \tag{213a}$$

$$\Gamma_\perp = \frac{d\zeta_\perp}{dt} + d\zeta_\perp \cdot \boldsymbol{V} - \boldsymbol{k}^t\left(\boldsymbol{V}, \boldsymbol{V}\right) . \tag{213b}$$

Proof:[5] First, we observe that contrary to motions for which $k=0$, as in the case of 3D elasticity, $\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}$ cannot be defined unambiguously, in general, for motions in an evolving ambient space when $k \geq 1$ similarly to the discussion in §3.2 on the computation of the shell acceleration. If $T_{(X,t)}\tilde{\varphi}$ is injective, then the implicit function theorem tells us that $\tilde{\varphi}$ is a local diffeomorphism at (X,t). In particular, one obtains a local vector field $\boldsymbol{\mathcal{V}}$ on $\mathcal{Q}$ in a neighborhood of $\tilde{\varphi}(X,t)$ such that $\boldsymbol{\mathcal{V}}(\tilde{\varphi}(X,t)) = \boldsymbol{\Upsilon}(X,t) = \boldsymbol{v}(\tilde{\varphi}(X,t),t)$ and we can define the material acceleration as

$$\boldsymbol{\Gamma}(X,t) = D^{\boldsymbol{h}}_{\tilde{\varphi}_X}\boldsymbol{\Upsilon}_X := \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}(\tilde{\varphi}(X,t)) .$$

Recall that we chose $\{\chi^\alpha\}_{\alpha=1,\ldots,n+k}$ to be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1,\ldots,\chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$, and such that the unit normal vector field $\boldsymbol{\eta}_i^t$ for $i \in \{1,\ldots,k\}$ is tangent to the coordinate curve χ^{n+i}. In this coordinate chart, we recall that $h_{\alpha(n+i)} = \left\langle\!\!\left\langle \tilde{\partial}_\alpha^t, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \delta_{\alpha(n+i)}$, which will be used frequently in the following computations. Note that when $T_{(X,t)}\tilde{\varphi}$ is injective, the set

[5] We also benefited from a discussion with Fabio Sozio.

of chosen unit normal vector fields $\{\boldsymbol{\eta}^t{}_i\}_{i=1,\ldots,k}$ is well defined on a neighborhood of $\tilde{\varphi}(X,t)$ in $\mathcal{Q}$. Hence, one can decompose $\boldsymbol{\mathcal{V}}$ into tangent and normal components with respect to $\{\boldsymbol{\eta}^t{}_i\}_{i=1,\ldots,k}$ as $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_\parallel + \boldsymbol{\mathcal{V}}_\perp$. One then writes

$$\boldsymbol{\Gamma}(X,t) = \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}(\boldsymbol{\mathcal{V}}_\parallel + \boldsymbol{\mathcal{V}}_\perp) = \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\parallel + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp\,.$$

Since the Levi-Civita connection is by definition torsion-free, one can write

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\parallel = \left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_\parallel\right] + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_\parallel}\boldsymbol{\mathcal{V}} = \left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_\parallel\right] + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_\parallel}\boldsymbol{\mathcal{V}}_\parallel + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_\parallel}\boldsymbol{\mathcal{V}}_\perp\,.$$

Note that since $\mathcal{V} = \mathcal{V}(\tilde{\varphi}(X,t))$ does not explicitly depend on time, one can write $\left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_\parallel\right] = \boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\parallel$, which is tangent to $\mathcal{S}_t$, where $\boldsymbol{L}$ denotes the Lie derivative.[6] Note that

$$\begin{aligned}
\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_\parallel}\boldsymbol{\mathcal{V}}_\parallel &= \psi_{t*}\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_\parallel}\psi_t^*\boldsymbol{\Upsilon}_\parallel - \sum_{i=1}^k \boldsymbol{\kappa}^t_i\left(\boldsymbol{\Upsilon}_\parallel, \boldsymbol{\Upsilon}_\parallel\right)\boldsymbol{\eta}^t_i\,,\\
\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_\parallel}\boldsymbol{\mathcal{V}}_\perp &= \sum_{i=1}^k \nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_\parallel}\left(\Upsilon^i_\perp\boldsymbol{\eta}^t_i\right) = \sum_{i=1}^k\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_\parallel}\Upsilon^i_\perp\right)\boldsymbol{\eta}^t_i + \sum_{i=1}^k \Upsilon^i_\perp\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_\parallel}\boldsymbol{\eta}^t_i\\
&= \sum_{i=1}^k\left(\tilde{d}\Upsilon^i_\perp\cdot\boldsymbol{\Upsilon}_\parallel\right)\boldsymbol{\eta}^t_i + \sum_{i=1}^k \Upsilon^i_\perp\boldsymbol{h}^\sharp\cdot\boldsymbol{\kappa}^t_i\cdot\boldsymbol{\Upsilon}_\parallel + \sum_{i,j=1}^k \Upsilon^i_\perp\left(\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{\Upsilon}_\parallel\right)\boldsymbol{\eta}^t_j\,,
\end{aligned}$$

where $\tilde{d}$ denotes the exterior derivative operator on $\mathcal{S}_t$, [7] and where we have used, following (258), that for $i \in \{1,\ldots,k\}$[8]

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_\parallel}\boldsymbol{\eta}^t_i = \boldsymbol{h}^\sharp\cdot\boldsymbol{\kappa}^t_i\cdot\boldsymbol{\Upsilon}_\parallel + \sum_{j=1}^k\left(\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{\Upsilon}_\parallel\right)\boldsymbol{\eta}^t_j\,.$$

Let us now compute $\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp$. We consider an arbitrary vector field $\boldsymbol{U}$ in $\mathcal{Q}$ such that $\boldsymbol{U}$ is tangent to $\mathcal{S}_t$ in a neighborhood of $\tilde{\varphi}(X,t)$, i.e., $\langle\!\langle\boldsymbol{\mathcal{V}}_\perp, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{h}} = 0$. Hence,

[6] The Lie derivative along the vector field $\boldsymbol{\mathcal{V}}$ is defined as $\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\parallel = \frac{d}{dt}\Big|_{t=s}\left[\left(\tilde{\varphi}_t\circ\tilde{\varphi}_s^{-1}\right)^*\boldsymbol{\mathcal{V}}_\parallel\right]$, where $\tilde{\varphi}_t\circ\tilde{\varphi}_s^{-1}$ is the flow of $\boldsymbol{\mathcal{V}}$.

[7] For a function f defined on $\mathcal{S}_t$, we write $\tilde{d}f = \sum_{\alpha=1}^n \frac{\partial f}{\partial\chi^\alpha}d\chi^\alpha$.

[8] Recall that the normal fundamental 1-forms are defined for $i,j \in \{1,\ldots,k\}$ as $\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{w} = \left\langle\!\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{w}}\boldsymbol{\eta}^t_i, \boldsymbol{\eta}^t_j\right\rangle\!\!\right\rangle_{\boldsymbol{h}}$ for any vector $\boldsymbol{w}$ tangent to $\mathcal{S}_t$.

$\left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp, \boldsymbol{U}\right\rangle\!\right\rangle_{\boldsymbol{h}} = -\left\langle\!\left\langle \boldsymbol{\mathcal{V}}_\perp, \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\right\rangle\!\right\rangle_{\boldsymbol{h}}$. However, at $\tilde{\varphi}(X,t)$, we have

$$
\begin{aligned}
\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U} &= [\boldsymbol{\mathcal{V}},\boldsymbol{U}] + \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}} = [\boldsymbol{\mathcal{V}},\boldsymbol{U}] + \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}}_\parallel + \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}}_\perp \\
&= [\boldsymbol{\mathcal{V}},\boldsymbol{U}] + \psi_{t*}\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{U}}\psi_t^*\boldsymbol{\Upsilon}_\parallel - \sum_{i=1}^{k}\boldsymbol{\kappa}^t_i\left(\boldsymbol{\Upsilon}_\parallel,\boldsymbol{U}\right)\boldsymbol{\eta}^t_i + \sum_{i=1}^{k}\left(\tilde{d}\Upsilon^i_\perp\cdot\boldsymbol{U}\right)\boldsymbol{\eta}^t_i \\
&+ \sum_{i=1}^{k}\Upsilon^i_\perp\nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\eta}^t_i\,.
\end{aligned}
$$

Thus, we have[9]

$$
\left\langle\!\left\langle \boldsymbol{\mathcal{V}}_\perp, \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\right\rangle\!\right\rangle_{\boldsymbol{h}} = -\sum_{i=1}^{k}\Upsilon^i_\perp\boldsymbol{\kappa}^t_i(\boldsymbol{\Upsilon}_\parallel,\boldsymbol{U}) + \sum_{i=1}^{k}\Upsilon^i_\perp\left(\tilde{d}\Upsilon^i_\perp\cdot\boldsymbol{U}\right) + \sum_{i,j=1}^{k}\Upsilon^i_\perp\Upsilon^j_\perp\left(\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{U}\right)\,,
$$

where we recall that $\left(\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{U}\right) = \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\eta}^t_i, \boldsymbol{\eta}^t_j\right\rangle\!\right\rangle_{\boldsymbol{h}}$. Therefore, it follows by arbitrariness of $\boldsymbol{U}$ from $\left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp, \boldsymbol{U}\right\rangle\!\right\rangle_{\boldsymbol{h}} = -\left\langle\!\left\langle \boldsymbol{\mathcal{V}}_\perp, \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\right\rangle\!\right\rangle_{\boldsymbol{h}}$ that

$$
\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp\right)_\parallel = \sum_{i=1}^{k}\Upsilon^i_\perp\boldsymbol{h}^\sharp\cdot\boldsymbol{\kappa}^t_i\cdot\boldsymbol{\Upsilon}_\parallel - \sum_{i=1}^{k}\Upsilon^i_\perp\left(\tilde{d}\Upsilon^i_\perp\right)^\sharp - \sum_{i,j=1}^{k}\Upsilon^i_\perp\Upsilon^j_\perp\boldsymbol{\omega}^{t\sharp}_{ij}\,.
$$

On the other hand, we have

$$
\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp = \sum_{i=1}^{k}\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\left(\mathcal{V}^i_\perp\boldsymbol{\eta}^t_i\right) = \sum_{i=1}^{k}\frac{d\mathcal{V}^i_\perp}{dt}\boldsymbol{\eta}^t_i + \sum_{i=1}^{k}\mathcal{V}^i_\perp\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_\parallel}{\boldsymbol{\eta}^t}_i + \sum_{j=1}^{k}\mathcal{V}^j_\perp\nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j}{\boldsymbol{\eta}^t}_i\right)\,.
$$

Then, it follows that at $\tilde{\varphi}(X,t)$, one can write

$$
\begin{aligned}
\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp\right)_\perp &= \sum_{i=1}^{k}\frac{d\Upsilon^i_\perp}{dt}\boldsymbol{\eta}^t_i + \sum_{i,j=1}^{k}\Upsilon^i_\perp\left(\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{\Upsilon}_\parallel\right){\boldsymbol{\eta}^t}_j + \sum_{i,j,l=1}^{k}\Upsilon^i_\perp\Upsilon^j_\perp\left\langle\!\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j}{\boldsymbol{\eta}^t}_i,\boldsymbol{\eta}^t_l\right\rangle\!\!\right\rangle_{\boldsymbol{h}}\boldsymbol{\eta}^t_l \\
&= \sum_{i=1}^{k}\left[\frac{d\Upsilon^i_\perp}{dt} + \sum_{j=1}^{k}\Upsilon^j_\perp\left(\boldsymbol{\omega}^t_{ji}\cdot\boldsymbol{\Upsilon}_\parallel\right) + \sum_{j,l=1}^{k}\Upsilon^j_\perp\Upsilon^l_\perp\left\langle\!\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j}{\boldsymbol{\eta}^t}_l,\boldsymbol{\eta}^t_i\right\rangle\!\!\right\rangle_{\boldsymbol{h}}\right]{\boldsymbol{\eta}^t}_i\,.
\end{aligned}
$$

[9]Note that since the vector $\boldsymbol{U}$ is tangent to $\mathcal{S}_t$ at $\tilde{\varphi}(X,t)$, the vector $[\boldsymbol{\mathcal{V}},\boldsymbol{U}] = \boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}$ is tangent to $\mathcal{S}_t$ as well.

Finally, the tangent and normal components of the acceleration vector read

$$\begin{aligned}\boldsymbol{\Gamma}_{\parallel} &= \boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} + \psi_{t*}\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_{\parallel}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} + 2\sum_{i=1}^{k}\Upsilon_{\perp}^i \boldsymbol{h}^{\sharp}\cdot\boldsymbol{\kappa}_i^t\cdot\boldsymbol{\Upsilon}_{\parallel} \\ &\quad - \sum_{i=1}^{k}\Upsilon_{\perp}^i\left(\tilde{d}\Upsilon_{\perp}^i\right)^{\sharp} - \sum_{i,j=1}^{k}\Upsilon_{\perp}^i\Upsilon_{\perp}^j\boldsymbol{\omega}_{ij}^{t\sharp}\,,\end{aligned} \tag{214a}$$

$$\boldsymbol{\Gamma}_{\perp} = \sum_{i=1}^{k}\left[\frac{d\Upsilon_{\perp}^i}{dt} + \tilde{d}\Upsilon_{\perp}^i\cdot\boldsymbol{\mathcal{V}}_{\parallel} - \boldsymbol{\kappa}_i^t\left(\boldsymbol{\Upsilon}_{\parallel},\boldsymbol{\Upsilon}_{\parallel}\right) + 2\sum_{j=1}^{k}\Upsilon_{\perp}^j\left(\boldsymbol{\omega}_{ji}^t\cdot\boldsymbol{\Upsilon}_{\parallel}\right)\right. \tag{214b}$$

$$\left. + \sum_{j,l=1}^{k}\Upsilon_{\perp}^j\Upsilon_{\perp}^l\left\langle\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_j^t}\boldsymbol{\eta}^t{}_l,\boldsymbol{\eta}_i^t\right\rangle\!\right\rangle_{\boldsymbol{h}}\right]\boldsymbol{\eta}_i^t\,. \tag{214c}$$

We recall that $\boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta}\circ\varphi_t\,,\ \boldsymbol{Z} := \psi_t^*\boldsymbol{\zeta}_{\parallel}\circ\varphi_t\,,\ \boldsymbol{A} := \psi_t^*\boldsymbol{\Gamma}_{\parallel}\,,\ \boldsymbol{k}_i^t = \psi_t^*\boldsymbol{\kappa}_i^t$ and $\boldsymbol{o}_{ij}^t = \psi_t^*\boldsymbol{\omega}_{ij}^t$. However, following [50, Theorem (6.19), p. 101], and recalling that $\boldsymbol{\mathcal{V}} = \boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta}$, one can write

$$\psi_t^*\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = \boldsymbol{L}_{\boldsymbol{V}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} = \boldsymbol{L}_{\boldsymbol{V}}\left(\boldsymbol{V}+\boldsymbol{Z}\right) = \frac{\partial}{\partial t}\left(V^a + Z^a\right)\partial_a + \left[\boldsymbol{V},\boldsymbol{V}+\boldsymbol{Z}\right]\,.$$

Note that since the connection is torsion-free, it follows that

$$\left[\boldsymbol{V},\boldsymbol{V}+\boldsymbol{Z}\right] = \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{V}\,.$$

Denoting by $D_t^{\boldsymbol{g}_t}$ the covariant derivative along the curve $t \to \varphi_t(X)$, one can write

$$D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) = \frac{\partial}{\partial t}\left(V^a + Z^a\right)\partial_a + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\left(\boldsymbol{V}+\boldsymbol{Z}\right)\,.$$

Therefore, one concludes that

$$\psi_t^*\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{V}\,.$$

We also have

$$\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_{\parallel}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} = \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\left(\boldsymbol{V}+\boldsymbol{Z}\right)\,.$$

Hence

$$\psi_t^*\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} + \nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_{\parallel}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} = \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\boldsymbol{Z} - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{Z}}\boldsymbol{V} + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\left(\boldsymbol{V}+\boldsymbol{Z}\right) = D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z}.$$

Therefore, denoting by d the exterior derivative on $\mathcal{S}$, [10] one can rewrite (214) as

$$\begin{aligned}\boldsymbol{A} = D_t^{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z}) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} + 2\sum_{i=1}^k \zeta_\perp^i \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}_i^t \cdot (\boldsymbol{V}+\boldsymbol{Z}) \\ - \sum_{i=1}^k \zeta_\perp^i \left(d\zeta_\perp^i\right)^\sharp - \sum_{i,j=1}^k \zeta_\perp^i\zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp}\,,\end{aligned} \tag{215a}$$

$$\begin{aligned}\Gamma_\perp^i = \frac{d\zeta_\perp^i}{dt} + d\zeta_\perp^i\cdot(\boldsymbol{V}+\boldsymbol{Z}) - \boldsymbol{k}_i^t(\boldsymbol{V}+\boldsymbol{Z},\boldsymbol{V}+\boldsymbol{Z}) - 2\sum_{j=1}^k \zeta_\perp^j \boldsymbol{o}_{ij}^t\cdot(\boldsymbol{V}+\boldsymbol{Z}) \\ + \sum_{j,l=1}^k \zeta_\perp^j\zeta_\perp^l \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_j^t}\boldsymbol{\eta}^t{}_l, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}}\,,\end{aligned} \tag{215b}$$

where $i=1,...,k$ and $d\zeta_\perp^i = \sum_{a=1}^n \frac{\partial\zeta_\perp^i}{\partial x^a}dx^a$. □

Let us next turn to the variation of the elastic energy, which is calculated as

$$\delta W = \frac{\partial W}{\partial \tilde{\boldsymbol{F}}} : \boldsymbol{L}_{\delta\tilde{\varphi}}\tilde{\boldsymbol{F}} + \frac{\partial W}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\delta\tilde{\varphi}}\boldsymbol{h}\,.$$

However, note that for an arbitrary time-independent material vector field $\boldsymbol{U}$, one has

$$\boldsymbol{L}_{\delta\tilde{\varphi}}\tilde{\boldsymbol{F}}\boldsymbol{U} = \left[\frac{d}{d\epsilon}\left(\tilde{\varphi}_\epsilon\circ\tilde{\varphi}_s^{-1}\right)^*\tilde{\boldsymbol{F}}\boldsymbol{U}\right]_{s=\epsilon} = \left[\frac{d}{d\epsilon}\tilde{\varphi}_{s*}\tilde{\varphi}_\epsilon^*\tilde{\varphi}_{\epsilon*}\boldsymbol{U}\right]_{s=\epsilon} = \left[\frac{d}{d\epsilon}\tilde{\varphi}_{s*}\boldsymbol{U}\right]_{s=\epsilon} = \boldsymbol{0}\,. \tag{216}$$

Thus, $\boldsymbol{L}_{\delta\tilde{\varphi}}\tilde{\boldsymbol{F}} = \boldsymbol{0}$. We also obtain, by using (266) and similarly to (267), that

$$\boldsymbol{L}_{\delta\tilde{\varphi}}\boldsymbol{h} = \mathfrak{L}_{\delta\tilde{\varphi}}\boldsymbol{h} = \psi_*\left(\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t + 2\sum_{i=1}^k \delta\tilde{\varphi}_\perp^i\boldsymbol{k}_i^t\right)\,, \tag{217}$$

where $\delta\tilde{\varphi}_\parallel$ is the part of $\delta\tilde{\varphi}$ tangent to $\psi_t(\mathcal{S})$, $\delta\tilde{\varphi}_\perp^i$, for $i\in\{1,...,k\}$, is its component along the unit normal $\boldsymbol{\eta}_i^t$, and $\mathfrak{L}$ denotes the autonomous Lie derivative. Therefore, recalling (206), it follows that

$$\delta W = \frac{\partial W}{\partial \boldsymbol{h}} : \left[\psi_*\left(\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t + 2\sum_{i=1}^k \delta\tilde{\varphi}_\perp^i\boldsymbol{k}_i^t\right)\right] = \frac{\partial W}{\partial \boldsymbol{g}} : \left(\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_\parallel}\boldsymbol{g}_t + 2\sum_{i=1}^k \delta\tilde{\varphi}_\perp^i\boldsymbol{k}_i^t\right)\,. \tag{218}$$

[10] For a function f defined on $\mathcal{S}$, we write $df = \sum_{a=1}^n \frac{\partial f}{\partial x^a}dx^a$.

Let us first assume that the variations of $\tilde{\varphi}$ are tangent to $\mathcal{S}_t$, i.e., $\delta\tilde{\varphi}^i_{\perp}=0\,,\forall i\in\{1,...,k\}$. Therefore, the variation of the action associated with the elastic energy reads

$$\delta S_W=-\int_{t_0}^{t_1}\int_{\mathcal{B}}\rho_0\frac{\partial W}{\partial\boldsymbol{g}}:\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_{\|}}\boldsymbol{g}_t dV dt\,.$$

Note, however, that $\mathfrak{L}_{\psi^*\delta\tilde{\varphi}_{\|}}\boldsymbol{g}_t=\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}^{\flat}_{\|}+\left[\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}^{\flat}_{\|}\right]^{\mathsf{T}}$. Hence, by symmetry of $\boldsymbol{g}_t$, one can write

$$\begin{aligned}
\delta S_W&=-\int_{t_0}^{t_1}\int_{\mathcal{B}}2\rho_0\frac{\partial W}{\partial\boldsymbol{g}}:\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}^{\flat}_{\|}dV dt\\
&=-\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}2\rho\frac{\partial W}{\partial\boldsymbol{g}}:\nabla^{\boldsymbol{g}_t}\psi^*\delta\tilde{\varphi}^{\flat}_{\|}dvdt\\
&=-\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}2\rho\frac{\partial W}{\partial g_{ab}}\left(\psi^*\delta\tilde{\varphi}_{\|}\right)_{a|b}dvdt\\
&=-\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial\boldsymbol{g}}\cdot\psi^*\delta\tilde{\varphi}^{\flat}_{\|}\right)dvdt\\
&\quad+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle\mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial\boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_{\|}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t}dvdt\\
&=-\int_{t_0}^{t_1}\int_{\partial\varphi_t(\mathcal{B})}\left(2\rho\frac{\partial W}{\partial\boldsymbol{g}}\cdot\psi^*\delta\tilde{\varphi}^{\flat}_{\|}\right)\cdot\mathbf{n}dadt\\
&\quad+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle\mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial\boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_{\|}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t}dvdt\,,
\end{aligned}$$

where ρ is the mass density in $\mathcal{S}$, $\mathrm{div}_{\boldsymbol{g}_t}$ (surface divergence) denotes the divergence operator in $(\mathcal{S},\boldsymbol{g}_t)$, and $\mathbf{n}$ is the unit normal vector to $\partial\varphi_t(\mathcal{B})$ in $\mathcal{S}$. Therefore, assuming that the variation of $\tilde{\varphi}$ is fixed on the boundary, i.e., $\delta\tilde{\varphi}|_{\partial\varphi(\mathcal{B})}=0$, one obtains

$$\delta S_W=\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle\mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial\boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_{\|}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t}dvdt\,.\tag{219}$$

Hence, by (209) and (219), the Lagrange-d'Alembert principle (207) reads

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle-\rho\boldsymbol{A}+\mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial\boldsymbol{g}}\right),\psi^*\delta\tilde{\varphi}_{\|}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t}dvdt\\
&\qquad+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\left\langle\rho\boldsymbol{B},\psi^*\delta\tilde{\varphi}_{\|}\right\rangle\!\right\rangle_{\boldsymbol{g}_t}dvdt=0\,,
\end{aligned}$$

where $\boldsymbol{B} = \psi^* \boldsymbol{\beta}_{\|}$. Therefore, by arbitrariness of $\delta\tilde{\varphi}_{\|}$, one obtains the following tangent Euler-Lagrange equations

$$\operatorname{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right) + \rho\boldsymbol{B} = \rho\boldsymbol{A}\,. \tag{220}$$

In terms of the Cauchy stress tensor $\boldsymbol{\sigma} = 2\rho\frac{\partial W}{\partial \boldsymbol{g}}$, we have

$$\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma} + \rho\boldsymbol{B} = \rho\boldsymbol{A}\,. \tag{221}$$

In the particular case when $\mathcal{S}_t$ is a hyperspace evolving transversally in $\mathcal{Q}$, i.e., $k=1$ and $\boldsymbol{Z}=\boldsymbol{0}$, the tangent Euler-Lagrange equations read

$$\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma} + \rho\boldsymbol{B} = \rho D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\rho\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot \boldsymbol{V} - \rho\zeta_\perp\left(d\zeta_\perp\right)^\sharp\,. \tag{222}$$

Equivalently, in terms of the rate of change of the spatial metric one has

$$\operatorname{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma} + \rho\boldsymbol{B} = \rho D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \rho\boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} - \rho\zeta_\perp\left(d\zeta_\perp\right)^\sharp\,. \tag{223}$$

Now, we assume that the variations of $\tilde{\varphi}$ are normal to $\mathcal{S}_t$, i.e., $\delta\tilde{\varphi}_{\|} = \boldsymbol{0}$. Using (209) and (218), we obtain from (207), by arbitrariness of $\delta\tilde{\varphi}_\perp^i$, the following normal Euler-Lagrange equations

$$-2\rho_0\frac{\partial W}{\partial \boldsymbol{g}}:\boldsymbol{k}_i^t + \rho_0 B_\perp^i = \rho_0\Gamma_\perp^i\,,\quad i=1,...,k\,. \tag{224}$$

In terms of the Cauchy stress, one has

$$-\boldsymbol{\sigma}:\boldsymbol{k}_i^t + \rho B_\perp^i = \rho\Gamma_\perp^i\,,\quad i=1,...,k\,. \tag{225}$$

Remark 5.1.3. Eq.(222) is identical to the tangential component of [75]'s Eq. (27), However, we believe that the expression of the acceleration he wrote before his Eq. (16) should be corrected to include the extra terms depending on the second fundamental form and the gradient of the embedding normal velocity as can be seen in (222). If one neglects the inertial terms, Eq.(222) is identical to [4]'s Eq. (4). However, it is not identical to their Eq. (3) in the presence of inertial effects. For them acceleration reads

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}} \boldsymbol{V} + V_n H \boldsymbol{V},$$

where $H = g^{ab} k_{ab}$ is twice the mean curvature. Note that, when $\boldsymbol{Z} = \boldsymbol{0}$ and $k = 1$, their $\boldsymbol{k}$ is our $-\boldsymbol{k}^t$, $V_n := \zeta_\perp$, and their acceleration should be corrected to read (in their notation)[11]

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}} \boldsymbol{V} - 2 V_n \boldsymbol{k} \cdot \boldsymbol{V} - V_n \left(dV_n \right)^\sharp .$$

We also note that in the case of a 2D shell embedded as a hypersurface in $\mathbb{R}^3$, (225) is identical to the normal component of [75]'s Eq. (27), although [75] did not write down the expression of the extrinsic acceleration. Ignoring the inertial terms, Eq. (225) is identical to [4]'s Eq. (5).

5.2 *Conservation of Mass for Motion in an Evolving Ambient Space*

Locally, conservation of mass is equivalent to

$$\rho(\boldsymbol{x}, t) J(\boldsymbol{X}, t) = \rho_0(\boldsymbol{X}),$$

where $J(\boldsymbol{X}, t) = \sqrt{\frac{\det \boldsymbol{g}_t}{\det \boldsymbol{G}}} \det \boldsymbol{F}$ is the Jacobian of deformation mapping φ.[12] Thus

$$\frac{d}{dt} (\rho J) = 0.$$

Note that

$$\dot{J} = \left(\operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{V} \right) J + \frac{1}{2} J \operatorname{tr} \left(\frac{\partial \boldsymbol{g}_t}{\partial t} \right) ,$$

where the superposed dot denotes total time differentiation, i.e., $\dot{J} = \frac{dJ}{dt}$. Therefore[13]

$$\dot{\rho} + \rho \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{V} + \frac{1}{2} \rho \operatorname{tr} \left(\frac{\partial \boldsymbol{g}_t}{\partial t} \right) = 0 . \tag{226}$$

[11] We communicated with A. DeSimone and M. Arroyo and they kindly confirmed the mistake in their acceleration. In their derivation they followed the *master balance law* of [50, p. 129].

[12] Note that the Jacobian of the deformation $\tilde{\varphi}$ is equal to that of φ, i.e., $\sqrt{\frac{\det \boldsymbol{h}}{\det \boldsymbol{G}}} \det \tilde{\boldsymbol{F}} = \sqrt{\frac{\det \boldsymbol{g}_t}{\det \boldsymbol{G}}} \det \boldsymbol{F}$, which follows from $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$.

[13] Note that there is a typo in the corresponding equation in [50, p. 92].

Note that even if $\boldsymbol{V} = \boldsymbol{0}$, ρ is time dependent. Therefore, in the case of a 2D shell transversally embedded in $\mathbb{R}^3$—recalling Lemma A.3.2, which in this case reads $\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\zeta_\perp \boldsymbol{k}^t + \nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^\mathsf{T}$—(226) can be written as

$$\dot{\rho} + \rho\, \mathrm{div}_{\boldsymbol{g}_t}(\boldsymbol{V} + \boldsymbol{Z}) + \rho\zeta_\perp H = 0\,, \tag{227}$$

where $H = \mathrm{tr}\, \boldsymbol{k}^t$ is twice the mean curvature. Eq. (227) is identical to the conservation of mass for shells appearing in [75, Eq. (21)], [50, p. 92], and [4, Eq. (1)]. Note that, if we look at the spatial mass density form $\boldsymbol{\rho} := \rho dv$, (226) reads

$$\boldsymbol{L}_{\boldsymbol{V}}\boldsymbol{\rho} = 0\,. \tag{228}$$

Equivalently, one can write

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{U})} \rho dv = \int_{\varphi_t(\mathcal{U})} \boldsymbol{L}_{\boldsymbol{V}}(\rho dv) = \int_{\varphi_t(\mathcal{U})} \left[\boldsymbol{L}_{\boldsymbol{V}}\rho\ dv + \rho \boldsymbol{L}_{\boldsymbol{V}}(dv)\right] = 0\,. \tag{229}$$

We know that

$$\boldsymbol{L}_{\boldsymbol{V}}(dv) = \mathfrak{L}_{\boldsymbol{V}}(dv) + \frac{\partial}{\partial t}(dv) = \left[\mathrm{div}\,\boldsymbol{V} + \frac{1}{2}\,\mathrm{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right)\right] dv. \tag{230}$$

Substituting (230) into (229) and localizing gives (226), which is the local form of conservation of mass.

5.3 Energy Balance in Nonlinear Elasticity in an Evolving Ambient Space

Let us consider an elastic body deforming in an evolving ambient space. We are interested in making an explicit connection between the deformation of the elastic body embedded in this ambient space and that in an ambient space with a dynamic metric. Let the ambient space $\mathcal{S}$ move in a larger (fixed) manifold $\mathcal{Q}$, i.e. $\psi_t : \mathcal{S} \to \mathcal{Q}$. The fixed background metric in $(\mathcal{Q}, \boldsymbol{h})$ induces a time-dependent metric on $\mathcal{S}$, i.e. $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$. Energy balance can be easily written in $\mathcal{Q}$ but we are interested to see how it is written for an observer in $\mathcal{S}$. When the metric $\boldsymbol{g}$ of $\mathcal{S}$ is fixed, the standard material balance of energy for a given subset $\mathcal{U} \subset \mathcal{B}$ reads

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{V}, \boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}}\right) dV = &\int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B}, \boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}} + R\right) dV \\ &+ \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}} + H\right) dA\,,\end{aligned} \tag{231}$$

where E, R, H, and $\boldsymbol{T}$ are the internal energy function per unit mass, the heat supply per unit mass, the heat flux per unit undeformed area, and the boundary traction vector per unit undeformed area, respectively. Note that $E = E(X, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g})$, where N is the specific entropy. However, here the ambient space is evolving in time and the energy balance should be modified to accommodate this time dependency. First, let us look at an example to motivate our discussion.

Example 5.3.1. Suppose the ambient space is a 2-dimensional sphere of radius R that shrinks/expands in time. Then, whatever elastic material lives on this sphere will be compressed/stretched over time. As a simple case, assume that the material manifold is also a sphere, with radius equal to the initial radius of the ambient sphere. Assume that the deformation map φ is constant over time, as the metric evolves. This means that there will be an increase in elastic energy over time, not accounted for in terms of the work done by external forces–since there are no external forces.

Let the ambient metric, as a function of time be $g_{ij}(\theta, \phi, t) = f(t) g_{ij}^{\text{sphere}}(R)(\theta, \phi)$, where t is time, $f(t)$ is some function of time (the shrinkage/expansion factor) such that $f(t) > 0$, $f(t_0) = 1$, and ${\boldsymbol{g}_t}^{\text{sphere}}(R)$ is the metric of the 2-sphere with radius R. Note that this is a uniform rescaling of the metric. Then, let the material manifold be just $G_{IJ}(\Theta, \Phi) = G_{IJ}^{\text{sphere}}(R)(\Theta, \Phi)$, and let the deformation map simply send Θ to θ and Φ to ϕ at all times. Therefore, even though the material "is not moving" in terms of the coordinates ϕ and θ (a given material point sits at the same ϕ and θ at all times), it is shrinking/expanding. Note that $\Psi = \Psi(\boldsymbol{X}, \boldsymbol{C})$, where $C_{AB} = F^a{}_A F^b{}_B g_{ab} f(t)$. Thus, even if $F^a{}_B = \delta^a{}_A$, we have $C_{AB} = \delta^a{}_A \delta^b{}_B g_{ab} f(t)$.

This means that Ψ explicitly depends on $f(t)$ and hence there is stored elastic energy coming from the changes in the ambient space metric.

To visualize the time dependency of the metric of the ambient space, let us embed the initial sphere of radius $r = R$ in the Euclidean space $\mathbb{R}^3$. We then assume that the ambient space moves in the Euclidean space, i.e. there is a map $\psi_t : S^2(r) \to \mathbb{R}^3$. Explicitly this can be written in the spherical coordinates as $(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = \psi_t(r, \theta, \phi) = (k(t)r, \theta, \phi)$ with $k(t) > 0$. Note that deformation mapping is the inclusion map, i.e. $(\theta, \phi) = \varphi_t(\Theta, \Phi) = (\Theta, \Phi)$. The metric of the Euclidean space in spherical coordinates reads $\boldsymbol{h} = \mathrm{diag}\left(1, \tilde{r}^2, \tilde{r}^2 \sin^2\theta\right)$. Now the map ψ_t induces a metric $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$ on the ambient space that has the following representation: $\boldsymbol{g}_t = \mathrm{diag}\left(k(t)^2 r^2, k(t)^2 r^2 \sin^2\theta\right)$. It is seen that $f(t) = k(t)^2$, i.e., when expanding the ambient space by k, all the square distances in the ambient space with the time-dependent metric are multiplied by $f = k^2$ as expected. It is seen that time dependency of the ambient space metric can be visualized using a time-dependent embedding in a larger space with a fixed background metric (see [89] for a similar discussion). In the following we look at this in the general case of an arbitrary deformable body.

Next, we prove the following proposition for an arbitrary deformable body:

Proposition 5.3.1. *Energy balance for a deformable body moving in an evolving ambient space is given by*

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t}\right) dV = \int_{\mathcal{U}} \rho_0 \Big[\langle\!\langle \boldsymbol{B} + \boldsymbol{F}_{\mathrm{fic}}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} + R \\ + \left(\frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2}\langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) \Big] dV + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}_t} + H\right) dA \,, \end{aligned} \tag{232}$$

where we recall that $E = E(\boldsymbol{X}, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t)$[14] *is the material internal energy density per unit mass,* N, R, H, $\boldsymbol{T}$, *and* $\boldsymbol{B}$ *are the specific entropy per unit mass, the heat*

[14]*Similar to the discussion of Remark 5.1.1, we can conclude that* $E(\boldsymbol{X}, \mathsf{N}, \psi_* \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{h}) = E(\boldsymbol{X}, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t)$.

supply per unit mass, the heat flux per unit undeformed area, the boundary traction vector per unit undeformed area, and the tangent body force per unit mass, respectively. We also recall that $\boldsymbol{V}$ *is the velocity of* φ_t *and* $\boldsymbol{Z} = \psi_t^* \boldsymbol{\zeta}_\parallel$ *is the tangent velocity of the embedding* ψ_t *.* $\boldsymbol{F}_{\text{fic}}$ *denotes the fictitious body force due to the evolution of* $\mathcal{S}_t$ *and reads*

$$\begin{aligned}
\boldsymbol{F}_{\text{fic}} &= -\left(\boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}\right) \\
&= -D_t^{\boldsymbol{g}_t}\boldsymbol{Z} - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} - 2\sum_{i=1}^{k}\zeta_\perp^i \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}_i^t \cdot (\boldsymbol{V}+\boldsymbol{Z}) + \sum_{i=1}^{k}\zeta_\perp^i \left(d\zeta_\perp^i\right)^\sharp + \sum_{i,j=1}^{k}\zeta_\perp^i\zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp} \\
&= -D_t^{\boldsymbol{g}_t}\boldsymbol{Z} + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}\right]^{\mathsf{T}} \cdot (\boldsymbol{V}+\boldsymbol{Z}) - \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot (\boldsymbol{V}+\boldsymbol{Z}) + \sum_{i=1}^{k}\zeta_\perp^i \left(d\zeta_\perp^i\right)^\sharp \\
&+ \sum_{i,j=1}^{k}\zeta_\perp^i\zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp} \,.
\end{aligned} \tag{233}$$

Note that in the particular case of a transversal evolution of the ambient space as a hyperspace in $\mathcal{Q}$ *, i.e.,* $\boldsymbol{Z} = \boldsymbol{0}$ *and* $k = 1$ *, the fictitious body force reduces to*

$$\boldsymbol{F}_{\text{fic}} = -2\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot \boldsymbol{V} + \zeta_\perp \left(d\zeta_\perp\right)^\sharp = -\boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} + \zeta_\perp \left(d\zeta_\perp\right)^\sharp \,. \tag{234}$$

Proof: For an observer in $\mathcal{Q}$, energy balance for a sub-body $\mathcal{U} \subset \mathcal{B}$ is written as

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}\right) dV = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\!\left\langle \tilde{\boldsymbol{\beta}}, \boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}} + R\right) dV + \\
\int_{\partial\mathcal{U}} \left(\left\langle\!\!\left\langle \tilde{\boldsymbol{T}}, \boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}} + H\right) dA \,.
\end{aligned}$$

Body force can be decomposed into tangent and normal components with respect to $\mathcal{S}_t$ as $\boldsymbol{\beta} = \boldsymbol{\beta}_\parallel + \sum_{i=1}^k B_\perp^i \boldsymbol{\eta}_i^t$. Note that the traction vector is tangent to $\mathcal{S}_t$. We denote $\boldsymbol{B} = \psi_t^*\boldsymbol{\beta}_\parallel$, and $\boldsymbol{T} = \psi_t^*\tilde{\boldsymbol{T}}$. Recalling that $\boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta} \circ \varphi_t$ and $\boldsymbol{Z} = \psi_t^*\boldsymbol{\zeta}_\parallel$, the energy balance is simplified to read

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}\right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B}, \boldsymbol{V}+\boldsymbol{Z}\rangle\!\rangle_{\boldsymbol{g}_t} + \sum_{i=1}^{k} B_\perp^i \zeta_\perp^i + R\right) dV \\
+ \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V}+\boldsymbol{Z}\rangle\!\rangle_{\boldsymbol{g}_t} + H\right) dA \,.
\end{aligned} \tag{235}$$

Note that

$$\frac{d}{dt}\frac{1}{2}\left\langle\!\left\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} = \left\langle\!\left\langle D_t^{\boldsymbol{h}} \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} = \left\langle\!\left\langle \boldsymbol{\Gamma}, \boldsymbol{\Upsilon} \right\rangle\!\right\rangle_{\boldsymbol{h}} = \left\langle\!\left\langle \boldsymbol{A}, \boldsymbol{V}+\boldsymbol{Z} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \sum_{i=1}^{k} \zeta_\perp^i \Gamma_\perp^i \,, \tag{236}$$

and

$$\frac{d}{dt}E = \boldsymbol{L}_{\boldsymbol{\Upsilon}} E = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \tilde{\boldsymbol{F}}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \tilde{\boldsymbol{F}} + \frac{\partial E}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} \,.$$

Similar to (216), we see that $\boldsymbol{L}_{\boldsymbol{\Upsilon}} \tilde{\boldsymbol{F}} = \boldsymbol{0}$. Note that[15]

$$\boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} = \boldsymbol{L}_{\psi_* \boldsymbol{V}} \boldsymbol{h} + \boldsymbol{L}_{\boldsymbol{\zeta}} \boldsymbol{h} = \mathfrak{L}_{\psi_* \boldsymbol{V}} \boldsymbol{h} + \psi_{t*} \frac{\partial \boldsymbol{g}_t}{\partial t} = \psi_* \left(\mathfrak{L}_{(\boldsymbol{V}+\boldsymbol{Z})} \boldsymbol{g}_t + \frac{\partial \boldsymbol{g}_t}{\partial t} \right) = \psi_* \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t \,,$$

where we used (265) to write $\boldsymbol{L}_{\boldsymbol{\zeta}} \boldsymbol{h} = \psi_{t*} \frac{\partial \boldsymbol{g}_t}{\partial t}$. Therefore, it follows that in $\mathcal{Q}$

$$\frac{dE}{dt} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t \,. \tag{237}$$

An observer in $\mathcal{S}$ writes the energy balance as

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} \right) dV = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\left\langle \boldsymbol{B}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \Xi + R \right) dV \\ + \int_{\partial\mathcal{U}} \left(\left\langle\!\left\langle \boldsymbol{T}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + H \right) dA \,, \end{aligned} \tag{238}$$

where $\Xi = 0$ if the ambient space metric is fixed. Note that in $(\mathcal{S}, \boldsymbol{g}_t)$, we have

$$\frac{d}{dt}\frac{1}{2}\left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} = \left\langle\!\left\langle D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \frac{1}{2}\left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} \,, \quad \text{and} \quad \frac{dE}{dt} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t \,. \tag{239}$$

Let us now find Ξ. Subtracting (238) from (235) and using (236), (237), and (239), one obtains

$$\begin{aligned} \int_{\mathcal{U}} \left(\rho_0 \left\langle\!\left\langle \boldsymbol{A}, \boldsymbol{Z} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} + \rho_0 \sum_{i=1}^{k} \zeta_\perp^i \Gamma_\perp^i \right) dV \\ = \int_{\mathcal{U}} \left(\rho_0 \left\langle\!\left\langle \boldsymbol{B}, \boldsymbol{Z} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \rho_0 \sum_{i=1}^{k} B_\perp^i \zeta_\perp^i - \Xi \right) dV + \int_{\partial\mathcal{U}} \left\langle\!\left\langle \boldsymbol{T}, \boldsymbol{Z} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} dA \,. \end{aligned}$$

[15] An alternate proof for this result can be found in [50, p. 101].

Note that $2\rho\dfrac{\partial E}{\partial \boldsymbol{g}} = 2\rho\dfrac{\partial W}{\partial \boldsymbol{g}} = \boldsymbol{\sigma}$, and $\dfrac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{Z}}\boldsymbol{g}_t = 2\dfrac{\partial E}{\partial \boldsymbol{g}} : \nabla^{\boldsymbol{g}_t}\boldsymbol{Z}$. Therefore, by using (220) and (224) we have

$$\int_{\mathcal{U}} \left(\rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) dV = \int_{\partial\mathcal{U}} \left\langle\!\left\langle \boldsymbol{T}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} dA$$
$$+ \int_{\mathcal{U}} \left(\sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{\sigma} : \boldsymbol{k}_i^t - J \left\langle\!\left\langle \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \Xi\right) dV .$$

Also, note that

$$\begin{aligned}
\int_{\partial\mathcal{U}} \left\langle\!\left\langle \boldsymbol{T}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} dA &= \int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \boldsymbol{\sigma}\cdot\boldsymbol{\eta}^t, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} da \\
&= \int_{\varphi_t(\mathcal{U})} \operatorname{div}_{\boldsymbol{g}_t} \left\langle\!\left\langle \boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} dv \\
&= \int_{\varphi_t(\mathcal{U})} \left(\boldsymbol{\sigma} : \nabla^{\boldsymbol{g}_t}\boldsymbol{Z} + \left\langle\!\left\langle \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t}\right) dv \\
&= \int_{\mathcal{U}} \left(\frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + J \left\langle\!\left\langle \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t}\right) dV .
\end{aligned}$$

Thus

$$\int_{\mathcal{U}} \left(\rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) dV =$$
$$\int_{\mathcal{U}} \left(\frac{1}{2}\boldsymbol{\sigma} : \left(\boldsymbol{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + 2\sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t\right) - \Xi\right) dV .$$

However, since $\boldsymbol{\sigma} = 2\rho\frac{\partial W}{\partial \boldsymbol{g}} = 2\rho\frac{\partial E}{\partial \boldsymbol{g}}$, and $\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t + \mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t$, it follows that

$$\Xi = \frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} - \rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \frac{1}{2}\rho_0 \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} .$$

Therefore, the balance of energy in $(\mathcal{S}, \boldsymbol{g}_t)$ reads

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t}\right) dV = \int_{\mathcal{U}} \rho_0 \Big(\left\langle\!\left\langle \boldsymbol{B} + \boldsymbol{F}_{\text{fic}}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t}$$
$$+ \frac{1}{2}\left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} + R\Big) dV + \int_{\partial\mathcal{U}} \left(\left\langle\!\left\langle \boldsymbol{T}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} + H\right) dA ,$$

where the fictitious body force $\boldsymbol{F}_{\text{fic}}$ is given in (233). If the evolution of the ambient space as a hyperspace in $\mathcal{Q}$ is transversal, i.e., $\boldsymbol{Z} = \boldsymbol{0}$ and $k = 1$, the fictitious body force reduces to (234). □

Remark 5.3.1. Note that the non-classical extra terms appearing in the energy balance (232) can be written as

$$\frac{\partial E}{\partial \boldsymbol{g}}:\frac{\partial \boldsymbol{g}_t}{\partial t}+\frac{1}{2}\langle\!\langle \boldsymbol{V},\boldsymbol{V}\rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}=\frac{\partial}{\partial \boldsymbol{g}}\left(E+\frac{1}{2}\langle\!\langle \boldsymbol{V},\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}_t}\right):\frac{\partial \boldsymbol{g}_t}{\partial t}\,,$$

so that this term cancels out the contribution of the rate of change of the energy (internal + kinetic) due to the evolving ambient space metric appearing on the left hand side of (232).

5.4 Quasi-Static Deformations of the Ambient Space Metric

Let us consider a spatial metric that depends on a position-dependent parameter $\omega(\boldsymbol{x})$, e.g. $\boldsymbol{g}=\boldsymbol{g}(\boldsymbol{x},\omega(\boldsymbol{x}))$. In other words, given an initial metric $\boldsymbol{g}_0$, we quasi-statically deform the ambient space manifold. As an application, we can think of a situation when a thin sheet of metal is compressed between two identical and evolving surfaces to make different curved sheets, e.g. some pieces of an automobile body. As an example, one can start with a rescaling of the spatial metric, i.e.

$$\boldsymbol{g}(\boldsymbol{x},\omega(\boldsymbol{x}))=e^{2\omega(\boldsymbol{x})}\boldsymbol{g}_0(\boldsymbol{x})\,.$$

Note that Jacobian J of deformation in the new ambient space is related to the Jacobian with respect to the old ambient space J_0 as follows:

$$J=\sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}}\det \boldsymbol{F}=e^{\frac{n\omega(\boldsymbol{x})}{2}}J_0\,,$$

where $\dim\mathcal{S}=n$. Having an equilibrium configuration, replacing $\boldsymbol{g}_0$ by its rescaled version, the equilibrium configuration will change, in general. The following two examples show the effect of a rescaling of the spatial metric on equilibrium configuration and the corresponding stresses.

We consider in this section a spherical cap embedded in a spherical ambient space. We uniformly rescale the spatial spherical metric (equivalently changing the radius

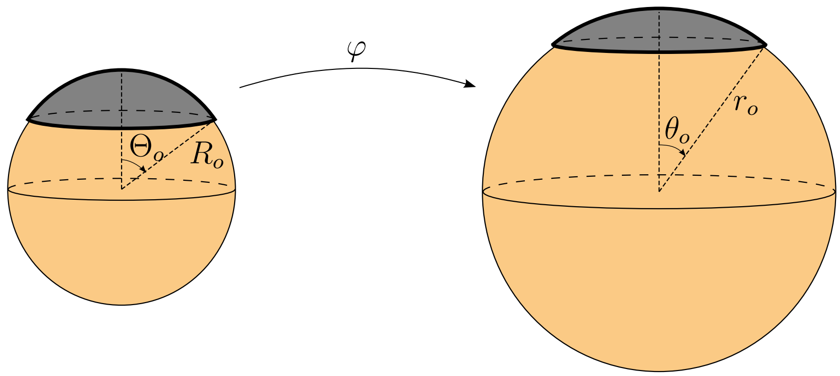


Figure 22: Deformation of a spherical cap due to a change in the radius of the ambient space.

of the sphere) and calculate the resulting stresses. We assume an incompressible and isotropic solid. For such solids the energy function depends on the first and second principal invariants of the right Cauchy-Green strain $\mathbf{C}$ (or the left Cauchy-Green strain $\mathbf{b}$, also known as the Finger deformation tensor), i.e. $W = W(I_1, I_2)$ [60]. Note that for an incompressible solid $I_3 = J^2 = 1$. The Finger deformation tensor $\boldsymbol{b}$ has components $b^{ab} = F^a{}_A F^b{}_B G^{AB}$. For an incompressible isotropic solid the Cauchy stress has the following representation [18, 84]

$$\boldsymbol{\sigma} = \left(-p + 2I_2 \frac{\partial W}{\partial I_2}\right) \boldsymbol{g}^\sharp + 2\frac{\partial W}{\partial I_1} \boldsymbol{b}^\sharp - 2\frac{\partial W}{\partial I_2} \boldsymbol{b}^{-1}, \tag{240}$$

where p is the Lagrange multiplier corresponding to the incompressibility constraint $J = 1$. Note that $\boldsymbol{b}^{-1} = \boldsymbol{c}$ has components $c^{ab} = \left(F^{-1}\right)^A{}_m \left(F^{-1}\right)^B{}_n \, G_{AB} \; g^{am} g^{bn}$.

Example 5.4.1. (Spherical cap on a 2D sphere) Let us consider a two-dimensional spherical cap of angular radius Θ_o lying on a sphere of initial radius R_o. We assume that the spherical cap is made of an incompressible and isotropic material. We would like to calculate the stresses that occur in the new equilibrium configuration after the radius of the ambient sphere is changed to r_o. See Fig. **??**. In spatial spherical coordinates (θ, ϕ), the spatial metric reads

$$\boldsymbol{g} = \begin{pmatrix} r_o^2 & 0 \\ 0 & r_o^2 \sin^2 \theta \end{pmatrix} .$$

Note that changing the radius of the sphere from R_o to r_o is equivalent to a uniform scaling of its spatial metric by $e^{2\omega_o} = \frac{r_o^2}{R_o^2}$. Changing the spatial metric the equilibrium configuration changes. We look for solutions of the form $\varphi(\Theta, \Phi) = (\theta, \phi) = (\theta(\Theta), \Phi)$. Thus, $\boldsymbol{F} = \mathrm{diag}(\theta'(\Theta), 1)$. It follows that the Jacobian reads

$$J = \theta'(\Theta) \frac{r_o^2 \sin\left[\theta(\Theta)\right]}{R_o^2 \sin \Theta} .$$

Assuming that the spherical cap is made of an incompressible material, i.e., $J = 1$, fixing rigid body translations by taking $\theta(0) = 0$, and since $0 \leq \theta < \pi$, we find that

$$\theta(\Theta) = \cos^{-1}\left[\frac{r_o^2}{R_o^2}(\cos\Theta - 1) + 1\right] . \tag{241}$$

For this deformation, the Finger tensor reads

$$\boldsymbol{b} = \begin{pmatrix} \frac{R_o^2 \sin^2(\Theta)}{r_o^4 \sin^2(\theta)} & 0 \\ 0 & \frac{1}{R_o^2 \sin^2(\Theta)} \end{pmatrix} ,$$

and hence $I_1 = \frac{R_o^2 \sin^2 \Theta}{r_o^2 \sin^2 \theta} + \frac{r_o^2 \sin^2 \theta}{R_o^2 \sin^2 \Theta}$ and $I_2 = 1$. Therefore, we obtain from (240) the nonzero stress components as

$$\begin{aligned} \sigma^{\theta\theta} &= -\frac{1}{r_o^2} p + \frac{R_o^2 \sin^2 \Theta}{r_o^4 \sin^2 \theta} \alpha + \frac{1}{r_o^2}\left(1 - \frac{r_o^2 \sin^2 \theta}{R_o^2 \sin^2 \Theta}\right) \beta , \\ \sigma^{\phi\phi} &= -\frac{1}{r_o^2 \sin^2 \theta} p + \frac{1}{R_o^2 \sin^2 \Theta} \alpha + \frac{1}{r_o^2 \sin^2 \theta}\left(1 - \frac{R_o^2 \sin^2 \Theta}{r_o^2 \sin^2 \theta}\right) \beta , \end{aligned} \tag{242}$$

where $p(\Theta)$ is the unknown Lagrange multiplier and

$$\alpha(\Theta) = 2\frac{\partial W(I_1, I_2)}{\partial I_1} , \quad \beta(\Theta) = 2\frac{\partial W(I_1, I_2)}{\partial I_2} .$$

Using (241), the physical components of stress (242) are written as

$$\begin{aligned} \hat{\sigma}^{\theta\theta} &= -p + \frac{r_o^2(\cos\Theta + 1)}{2r_o^2 + R_o^2 \cos\Theta - R_o^2} \alpha + \left[1 - \frac{2r_o^2 + R_o^2 \cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)}\right] \beta , \\ \hat{\sigma}^{\phi\phi} &= -p + \frac{2r_o^2 + R_o^2 \cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)} \alpha + \left[1 - \frac{r_o^2(\cos\Theta + 1)}{2r_o^2 + R_o^2 \cos\Theta - R_o^2}\right] \beta . \end{aligned} \tag{243}$$

In terms of the Cauchy stress tensor, the only non-trivial intrinsic equilibrium equation is ${\sigma^{\theta a}}_{|a}=0$, which reads

$$\frac{r_o^2\sin^2\theta}{R_o^2\sin^2\Theta}{\sigma^{\theta\theta}}_{,\Theta}+\frac{1}{\tan\theta}\sigma^{\theta\theta}-\sin\theta\cos\theta\sigma^{\phi\phi}=0\,. \tag{244}$$

By using (241) and (242), the equilibrium equation (244) reduces to

$$\begin{aligned}p'=&-\tan^2\left(\frac{\Theta}{2}\right)\beta'+\frac{r_o^2\alpha'}{R_o^2}-\frac{\tan\left(\frac{\Theta}{2}\right)\left(r_o^2-2R_o^2\right)(\alpha+\beta)}{2r_o^2}\\&+\frac{r_o^2\sin(\Theta)(r_o^2-R_o^2)(\beta-\alpha)}{(2r_o^2+R_o^2\cos\Theta-R_o^2)^2}+\frac{R_o^2\tan^2\left(\frac{\Theta}{2}\right)\beta'}{r_o^2}+\frac{8\sin^4\left(\frac{\Theta}{2}\right)\left(R_o^2-r_o^2\right)\beta}{r_o^2\sin^3\Theta}\\&+\frac{4r_o^2\left(R_o^2-r_o^2\right)\alpha'+R_o^2\sin(\Theta)\left(R_o^2-2r_o^2\right)(\alpha+\beta)}{4r_o^2R_o^2+2R_o^4\cos\Theta-2R_o^4}\,.\end{aligned} \tag{245}$$

Note that, unlike the previous example, the spherical cap does not necessarily remain stress-free by a uniform scaling of the spatial metric. If, however, we take $r_o=R_o$, then (245) reduce to $p'=\alpha'$, which yields no stress by assuming zero boundary traction at $\Theta=\Theta_o$. Hence, we recover the case of a trivial embedding. Back to the general case when $r_o\neq R_o$, the evolving ambient sphere can be isometrically embedded in $\mathbb{R}^3$, i.e., $\mathcal{Q}=\mathbb{R}^3$, where the second fundamental form of the sphere reads

$$\boldsymbol{k}=\begin{pmatrix}-r_o & 0\\ 0 & -r_o\sin^2\theta\end{pmatrix}.$$

We only have one extrinsic equilibrium equation (225), which gives the normal component of the body force required to balance the stress field in the spherical cap. It is written as

$$B^n=\frac{1}{\rho}\boldsymbol{\sigma}:\boldsymbol{k}=-\frac{\hat{\sigma}^{\theta\theta}+\hat{\sigma}^{\phi\phi}}{r_o\rho}\,. \tag{246}$$

In the following, we explore the particular case when the spherical cap is made of a neo-Hookean solid, i.e., $\alpha(R)=\mu$ and $\beta(R)=0$. For a neo-Hookean solid, (245) reduces to

$$p'=\mu\frac{\tan\left(\frac{\Theta}{2}\right)\left(2R_o^2-r_o^2\right)}{2r_o^2}-\mu\frac{r_o^2\sin(\Theta)(r_o^2-R_o^2)}{(2r_o^2+R_o^2\cos(\Theta)-R_o^2)^2}+\mu\frac{\sin(\Theta)\left(R_o^2-2r_o^2\right)}{4r_o^2+2R_o^2\cos(\Theta)-2R_o^2}\,. \tag{247}$$

Therefore, assuming zero boundary traction at $\Theta = \Theta_o$, i.e., $\sigma^{\theta\theta}(\Theta_o) = 0$, we find that

$$p(\Theta) = \mu \left[g(\Theta) - g(\Theta_o) + \frac{r_o^2(\cos\Theta_o + 1)}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} \right], \tag{248}$$

where

$$\begin{aligned} g(\Theta) = \frac{2r_o^2 - R_o^2}{2R_o^2} \log\left(2r_o^2 + R_o^2\cos\Theta - R_o^2\right) + \frac{2R_o^2 - r_o^2}{2r_o^2} \log\left[\cos^2\left(\frac{\Theta}{2}\right)\right] \\ - \frac{r_o^2\left(r_o^2 - R_o^2\right)}{R_o^2\left(2r_o^2 + R_o^2\cos\Theta - R_o^2\right)}. \end{aligned} \tag{249}$$

Therefore, the stress field (243) and the extrinsic body force (246) are given by

$$\begin{aligned} \hat{\sigma}^{\theta\theta}(\Theta) &= \mu \Bigg[\frac{R_o^2 - 2r_o^2}{2R_o^2} \log\left(\frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2}\right) + \frac{r_o^2\cos\Theta}{2r_o^2 + R_o^2\cos\Theta - R_o^2} \\ &\quad - \frac{r_o^2\cos\Theta_o}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} + \frac{r_o^2 - 2R_o^2}{2r_o^2} \log\left(\frac{1 + \cos\Theta_o}{1 + \cos\Theta}\right) \\ &\quad - \frac{r_o^4(\cos\Theta - \cos\Theta_o)}{\left(2r_o^2 + R_o^2\cos\Theta - R_o^2\right)\left(2r_o^2 + R_o^2\cos\Theta_o - R_o^2\right)} \Bigg] \\ \hat{\sigma}^{\phi\phi}(\Theta) &= \mu \Bigg[\frac{R_o^2 - 2r_o^2}{2R_o^2} \log\left(\frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2}\right) + \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)} \\ &\quad - \frac{r_o^2(\cos\Theta_o + 1)}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} + \frac{r_o^2 - 2R_o^2}{2r_o^2} \log\left(\frac{1 + \cos\Theta_o}{1 + \cos\Theta}\right) \\ &\quad - \frac{r_o^2(r_o^2 - R_o^2)(\cos\Theta - \cos\Theta_o)}{\left(2r_o^2 + R_o^2\cos\Theta - R_o^2\right)\left(2r_o^2 + R_o^2\cos\Theta_o - R_o^2\right)} \Bigg] \\ r_o\rho B^n(\Theta) &= \mu \Bigg[\frac{2r_o^2 - R_o^2}{R_o^2} \log\left(\frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2}\right) + \frac{r_o^2(\cos\Theta_o + 1)}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} \\ &\quad - \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)} + \frac{2R_o^2 - r_o^2}{r_o^2} \log\left(\frac{1 + \cos\Theta_o}{1 + \cos\Theta}\right) \Bigg]. \end{aligned} \tag{250}$$

We plot in Fig. **??** the profile of stresses and the extrinsic body force in a spherical cap of angular radius $\Theta_o = \frac{\pi}{4}$ initially lying on a sphere of radius R_o, due to a change of the radius of the ambient sphere to $r_o = 1.5R_o$.

In the limiting case $r_o \to \infty$, which corresponds to flattening the spherical cap, we obtain from (251) that the stress field is given by

$$\begin{aligned} \hat{\sigma}^{\theta\theta}(\Theta) &= -\mu \left[\frac{1}{2} \log\left(\frac{1 + \cos\Theta}{1 + \cos\Theta_o}\right) + \frac{1}{4}(\cos\Theta - \cos\Theta_o) \right], \\ \hat{\sigma}^{\phi\phi}(\Theta) &= -\mu \left[\frac{1}{2} \log\left(\frac{1 + \cos\Theta}{1 + \cos\Theta_o}\right) + \frac{3}{4}(\cos\Theta - \cos\Theta_o) - \frac{2}{\cos\Theta + 1} + \frac{\cos\Theta_o + 1}{2} \right]. \end{aligned} \tag{251}$$

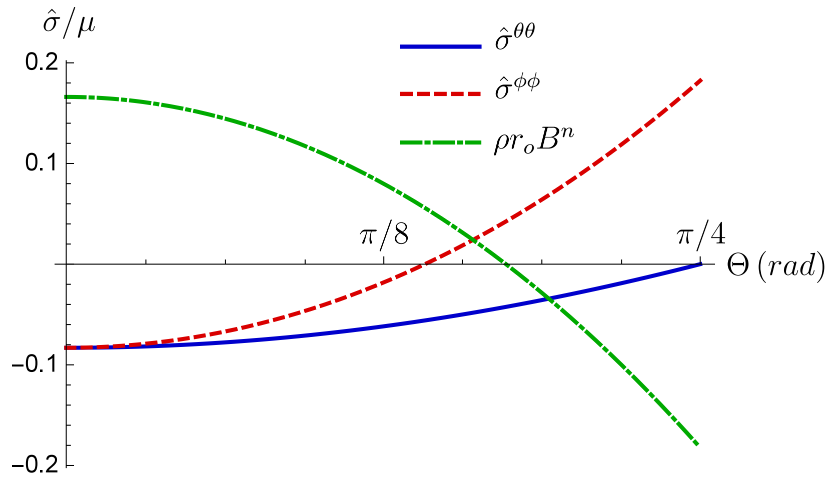


Figure 23: Stresses and the extrinsic body force in a spherical cap of initial angular radius $\Theta_o = \frac{\pi}{4}$ initially lying on a sphere of radius R_o, due to a change of the radius of the ambient sphere to $r_o = 1.5R_o$.

Note that the extrinsic body force field B^n vanishes when $r_o \to \infty$, since this case corresponds to a flat geometry of the ambient space.

CHAPTER VI

CONCLUDING REMARKS

In this PhD thesis, we present a geometric framework for referential and spatial evolutions in nonlinear elasticity. We use the concept of referential evolution to formulate a general theory of anelasticity and that of spatial evolution to study the motion of a continuum in a deforming ambient space.

First, we present a theory of anelasticity for three-dimensional bodies. In this theory, the reference configuration is an abstract manifold endowed with a time dependent Riemannian metric constructed to take into account the eigenstrain distribution in the body such that the material manifold is stress-free. The motion governing equations are derived using the Lagrange-d'Alembert variational principle for dissipative processes. By further taking the material metric as a dynamical variable, the variational formulation yields a kinetic equation governing the evolution of the material geometry. We apply this theory to thermoelasticity to show in a practical case how such a material metric is constructed to ensure that the evolving reference configuration remains stress-free. In order to derive the generalized heat equation governing the evolution of the referential geometry for thermoelasticity, we revisit the first and second laws of thermodynamics in our framework where additional terms must be included to account for the dynamic nature of the material geometry. As an example of a thermoelastic problem, we consider a spherical ball with a spherically-symmetric temperature distribution and solve for the thermal stress field as well as their evolution when the sphere is initially subject to a spherical thermal inclusion.

Second, we present of theory of shell anelasticity. In this theory, the idealized thin body is modeled as a shell and its midsurface is identified with a two-dimensional hypersurface embedded in the three-dimensional thin body. The anelastic eigenstrains are modeled using an evolving intrinsic geometry of the hypersurface, i.e., time-dependent first and second fundamental forms, so that the material shell is stress-free. The first and second fundamental forms are used to define intrinsic measures of strain to quantify the in-plane and out-of-plane deformations. By using a variational formulation following the Lagrange-d'Alembert principle, the motion governing equations are obtained in terms of the stress and couple-stress tensors which are naturally defined as conjugate to the aforementioned intrinsic strain measures. The nonlinear shell compatibility equations were related to the Gauss and Codazzi-Mainardi equations and lead to a systematic method to find stress-free anelastic distributions in simply-connected shells. The kinetic equations governing the evolution of the material shell geometry are obtained by considering the first and second fundamental forms as dynamical variable in the variational principle. The anelastic shell theory is applied to the case of morphoelastic shells, i.e., growing shells subject to bulk growth and remodeling. As an example, we considered a planar sheet and found a family of stress-free growth fields. We observed that stress-free growth can evolve a planar shell into another flat shell, a positively curved, or a negatively curved one. We also considered a growing circular shell that evolves to a curved cap and found the induced residual stresses and couple-stresses.

Third, we introduce a geometric theory of small-on-large anelasticity to study the induced small deformations due to a perturbation of a given distribution of (finite) eigenstrains superposed on the finite deformation that corresponds to the original distribution. Such an approach can be used to extend the class of known exact solutions in anelasticity beyond the highly symmetric ones based on semi-inverse methods. Given a nonlinear solid with a given distribution of eigenstrains, a perturbation of the eigenstrains changes the equilibrium configuration and its state of stress. In the geometric formulation of anelasticity, a perturbation of the anelasticity source corresponds to a perturbation of the geometry of the material manifold. We find the incremental residual stresses due to the perturbation fields and derive the governing equations for the induced small deformations superposed on the original finite deformation. To illustrate the capability of the theory, we consider an axi-symmetric distribution of parallel screw dislocations in an incompressible isotropic solid and calculate the perturbation fields when the body undergoes an arbitrary small perturbation in the Burgers' vector distribution. For generalized neo-Hookean solids, we are able to reduce the governing equations to a single ordinary differential equation for which we find a closed-form solution when the solid is neo-Hookean.

Finally, we formulate a theory of nonlinear elasticity with a deforming ambient space. The spatial evolution is modeled by considering a space-time embedding of the ambient space in a higher dimensional space with a fixed metric. Starting from a Lagrangian field theory considering the fixed background space, we reduce the variational formulation for the evolving ambient space and derive tangential and normal governing equations of motion. We show that the energy balance must be modified to include the time-dependency of the spatial geometry and reduce to that written by an observer in the evolving ambient space. As an example, we find a closed-form solution for a spherical cap deforming on a quasi-statically evolving sphere.

As a sequel to this work, we see many lines of investigations that we would like to explore. One can readily see the benefit of formulating a theory of small-on-large anelasticity for shells. Similarly to the three-dimensional theory, it would allow to extend the limited classes of known exact solutions for anelastic shells. Besides, further developments are needed in the theory of small-on-large anelasticity in order to introduce a systematic approach to investigate the stability of eigenstrain distributions (such as defects) in solids. Another line of investigation is the formulation of theory of elasticity with a deforming ambient space which includes couple stresses.

It would allow for example to consider both membranar and bending effects in shells deforming on a curved dynamical surface.

APPENDIX A

ELEMENTS OF DIFFERENTIAL GEOMETRY

A.1 Riemannian Geometry

Let us tersely review in the following a few elements and concepts of differential geometry that we use in this work. For more details, see for example [17]. Let $\mathcal{B}$ be an n-dimensional smooth manifold. For each $p \in \mathcal{B}$, a metric $\boldsymbol{G}$ for M gives a symmetric, positive definite, and bilinear smooth map $\boldsymbol{G}_p : T_p\mathcal{B} \times T_p\mathcal{B} \to \mathbb{R}$, where $T_p\mathcal{B}$ denotes the tangent space of $\mathcal{B}$ at p, such that for any smooth vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ on $\mathcal{B}$, $p \to \boldsymbol{G}_p(X_p, Y_p)$ is a smooth map.

A linear connection on a manifold $\mathcal{B}$ is a map $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathcal{X}(\mathcal{B}), \forall f, g \in C^\infty(\mathcal{B})$, where $C^\infty(\mathcal{B})$ is the set of smooth maps on $\mathcal{B}$ to $\mathbb{R}$, we have

$$\nabla_{\boldsymbol{X}}(\boldsymbol{Y} + \boldsymbol{Z}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \nabla_{\boldsymbol{X}}\boldsymbol{Z}\,,$$

$$\nabla_{f\boldsymbol{X}+g\boldsymbol{Y}}\boldsymbol{Z} = f\nabla_{\boldsymbol{X}}\boldsymbol{Z} + g\nabla_{\boldsymbol{Y}}\boldsymbol{Z}\,,$$

$$\nabla_{\boldsymbol{X}}(f\boldsymbol{Y}) = f\nabla_{\boldsymbol{X}}\boldsymbol{Y} + (\boldsymbol{X}f)\boldsymbol{Y}\,.$$

The vector field $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ is called the covariant derivative of $\boldsymbol{Y}$ along $\boldsymbol{X}$. In a local chart $\{X^A\}$, we have $\nabla_{\partial_A}\partial_B \in \mathcal{X}(\mathcal{B})$, and hence there exist scalars $\Gamma^C{}_{AB}$, called the Christoffel symbol of the connection, such that $\nabla_{\partial_A}\partial_B = \Gamma^C{}_{AB}\partial_C$. A linear connection is said to be compatible with a metric $\boldsymbol{G}$ on the manifold if

$$\boldsymbol{X}\left(\boldsymbol{G}\left(\boldsymbol{Y}, \boldsymbol{Z}\right)\right) = \boldsymbol{G}\left(\nabla_{\boldsymbol{X}}\boldsymbol{Y}\,, \boldsymbol{Z}\right) + \boldsymbol{G}\left(\boldsymbol{Y}\,, \nabla_{\boldsymbol{X}}\boldsymbol{Z}\right)\,.$$

It can be shown that ∇ is compatible with $\boldsymbol{G}$ if and only if $\nabla\boldsymbol{G} = \boldsymbol{0}$, which in components reads

$$G_{AB|C} = \frac{\partial G_{AB}}{\partial X^C} - \Gamma^K{}_{CA}G_{KB} - \Gamma^K{}_{CB}G_{AK} = 0\,.$$

We define the Lie bracket of two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ as the vector denoted by $[\boldsymbol{X},\boldsymbol{Y}]$ such that $\forall f \in C^\infty(\mathcal{B})$, we have

$$[\boldsymbol{X},\boldsymbol{Y}]f = \boldsymbol{X}\left(\boldsymbol{Y}f\right) - \boldsymbol{Y}\left(\boldsymbol{X}f\right) .$$

The torsion of a connection is defined as

$$\boldsymbol{T}(\boldsymbol{X},\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X},\boldsymbol{Y}] .$$

In components $T^A{}_{BC} = \Gamma^A{}_{BC} - \Gamma^A{}_{CB}$. ∇ is said to be symmetric if it is torsion-free, i.e., $\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} = [\boldsymbol{X},\boldsymbol{Y}]$. It can be shown that on any manifold $(\mathcal{B},\boldsymbol{G})$ there is a unique linear connection ∇ that is both compatible with $\boldsymbol{G}$ and is torsion-free. This result is the fundamental theorem of Riemannian geometry and such a connection is called the Levi-Civita connection. When endowed with its Levi-Civita connection, the pair $(\mathcal{B},\boldsymbol{G})$ is called a Riemannian manifold. It can be shown that the Christoffel symbol of the Levi-Civita connection associated with the metric $\boldsymbol{G}$ reads

$$\Gamma^A{}_{BC} = \frac{1}{2}\sum_K G^{AK}\left(\partial_C G_{KB} + \partial_B G_{KC} - \partial_K G_{BC}\right) .$$

The curvature tensor $\boldsymbol{\mathcal{R}}$ of a manifold $(\mathcal{B},\boldsymbol{G},\nabla)$ is given by[1]

$$\boldsymbol{\mathcal{R}}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} ,$$

for $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z} \in \mathcal{X}(M)$. In components

$$\mathcal{R}^A{}_{BCD} = dX^A(\boldsymbol{\mathcal{R}}(\partial_C,\partial_D)\partial_B) = \frac{\partial\Gamma^A{}_{DB}}{\partial X^C} - \frac{\partial\Gamma^A{}_{CB}}{\partial X^D} + \Gamma^A{}_{CK}\Gamma^K{}_{DB} - \Gamma^A{}_{DK}\Gamma^K{}_{CB} .$$

A.2 Geometry of an embedded hypersurface

In this section, we tersely review some elements of the geometry of two-dimensional embedded surfaces in three-dimensional manifolds. Let $\left(\mathcal{B},\bar{\boldsymbol{G}}\right)$ be an orientable three-dimensional Riemannian manifold and let $(\mathcal{H},\boldsymbol{G})$ be an orientable two-dimensional

[1] For a Riemannian manifold $(\mathcal{B},\boldsymbol{G})$, the curvature tensor is given in terms of its Levi-Civita connection.

Riemannian submanifold of $\left(\mathcal{B},\bar{\boldsymbol{G}}\right)$, i.e., $\boldsymbol{G}=\bar{\boldsymbol{G}}|_{\mathcal{H}}$. Let $\mathfrak{X}(\mathcal{H})$ be the space of smooth tangent vector fields on $\mathcal{H}$. Using the decomposition $T_X\mathcal{B}=T_X\mathcal{H}\oplus(T_X\mathcal{H})^{\perp}$, $\forall X\in\mathcal{H}$, we define the space of smooth normal vector fields $\mathfrak{X}(\mathcal{H})^{\perp}\subset\mathfrak{X}(\mathcal{B})$. Let $\boldsymbol{N}\in\mathfrak{X}(\mathcal{H})^{\perp}$ be the smooth unit normal vector field of $\mathcal{H}$. The orientation of the unit normal vector field $\boldsymbol{N}$ is chosen such that the orientation induced by the local coordinate chart of the surface $\mathcal{H}$ and the unit normal vector field as the last coordinate on $\mathcal{B}$ is consistent with the orientation of $\mathcal{B}$. Let $\nabla^{\mathcal{H}}$ and $\bar{\nabla}$ be the Levi-Civita connections of $(\mathcal{H},\boldsymbol{G})$ and $\left(\mathcal{B},\bar{\boldsymbol{G}}\right)$, respectively. Note that the Levi-Civita connection $\nabla^{\mathcal{H}}$ of the metric $\boldsymbol{G}$ is precisely the connection induced by the Levi-Civita connection $\bar{\nabla}$ of the metric $\bar{\boldsymbol{G}}$. The connection $\nabla^{\mathcal{H}}$ in terms of the connection $\bar{\nabla}$ is given by

$$\nabla^{\mathcal{H}}_{\boldsymbol{X}}\boldsymbol{Y}=\bar{\nabla}_{\bar{\boldsymbol{X}}}\bar{\boldsymbol{Y}}-\bar{\boldsymbol{G}}\left(\bar{\nabla}_{\bar{\boldsymbol{X}}}\bar{\boldsymbol{Y}},\boldsymbol{N}\right)\boldsymbol{N}\,,\ \forall\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{X}(\mathcal{H})\,,$$

where $\bar{\boldsymbol{X}}\in\mathfrak{X}(\mathcal{B})$ and $\bar{\boldsymbol{Y}}\in\mathfrak{X}(\mathcal{B})$ are any local extensions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, i.e., $\bar{\boldsymbol{X}}(X)=\boldsymbol{X}(X)$, $\forall X\in\mathcal{H}$. The second fundamental form of $\mathcal{H}$ is defined as the symmetric tensor $\boldsymbol{B}\in\Gamma(S^2T^*\mathcal{H})$ given by

$$\boldsymbol{B}(\boldsymbol{X},\boldsymbol{Y})=\bar{\boldsymbol{G}}\left(\bar{\nabla}_{\bar{\boldsymbol{X}}}\bar{\boldsymbol{Y}},\boldsymbol{N}\right)=-\bar{\boldsymbol{G}}\left(\bar{\nabla}_{\bar{\boldsymbol{X}}}\boldsymbol{N},\bar{\boldsymbol{Y}}\right)\,,\ \forall\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{X}(\mathcal{H})\,. \tag{252}$$

The connection ∇ on $T\mathcal{H}$ induces a connection on $S^2T^*\mathcal{H}$ defined by

$$\left(\nabla_{\boldsymbol{X}}\boldsymbol{A}\right)(\boldsymbol{Y},\boldsymbol{Z})=\boldsymbol{X}\left(\boldsymbol{A}(\boldsymbol{Y},\boldsymbol{Z})\right)-\boldsymbol{A}(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z})-\boldsymbol{A}(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z})\,,\quad\forall\boldsymbol{A}\in\Gamma(S^2T^*\mathcal{H})\,.$$

The curvature tensor $\boldsymbol{\mathcal{R}}$ of a Riemannian manifold $(M,\boldsymbol{G})$ is defined as

$$\boldsymbol{\mathcal{R}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W})=\boldsymbol{G}(\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z},\boldsymbol{W})\,,\ \forall\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}\in\mathfrak{X}(\mathcal{M})\,,$$

where $\boldsymbol{R}$ is given in terms of the Levi-Civita connection ∇^M by

$$\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}=\nabla^M_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}-\nabla^M_{\boldsymbol{X}}\nabla^M_{\boldsymbol{Y}}\boldsymbol{Z}+\nabla^M_{\boldsymbol{Y}}\nabla^M_{\boldsymbol{X}}\boldsymbol{Z}\,.$$

In components, the curvature tensor reads

$$\mathcal{R}_{ABCD}=\boldsymbol{\mathcal{R}}(\partial_A,\partial_B,\partial_C,\partial_D)=\left(\partial_B\Gamma^K{}_{AC}-\partial_A\Gamma^K{}_{BC}+\Gamma^L{}_{AC}\Gamma^K{}_{BL}-\Gamma^L{}_{BC}\Gamma^K{}_{AL}\right)G_{KD}\,.$$

Given the symmetries of the curvature tensor, if n is the dimension of the manifold M, its curvature tensor $\boldsymbol{\mathcal{R}}$ has $n^2(n^2-1)/12$ independent components. In particular, for a two-dimensional surface ($n = 2$), the curvature tensor has one independent component $\mathcal{R}_{1221}$.

We denote the Riemann curvature tensors of $\mathcal{H}$ and $\mathcal{B}$ by $\boldsymbol{\mathcal{R}}^{\mathcal{H}}$ and $\bar{\boldsymbol{\mathcal{R}}}$, respectively. The Gauss equation gives a relation between the Riemann curvature tensor and the second fundamental form of $\mathcal{H}$, and the Riemann curvature tensor of $\mathcal{B}$ as

$$\bar{\boldsymbol{\mathcal{R}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = \boldsymbol{\mathcal{R}}^{\mathcal{H}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) - \boldsymbol{B}(\boldsymbol{X},\boldsymbol{Z})\,\boldsymbol{B}(\boldsymbol{Y},\boldsymbol{W}) + \boldsymbol{B}(\boldsymbol{X},\boldsymbol{W})\,\boldsymbol{B}(\boldsymbol{Y},\boldsymbol{Z})\,. \tag{253}$$

The second fundamental form also satisfies the Codazzi-Mainardi equation that can be written as

$$\bar{\boldsymbol{\mathcal{R}}}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{N}) = \left(\nabla^{\mathcal{H}}_{\boldsymbol{Y}}\boldsymbol{B}\right)(\boldsymbol{X},\boldsymbol{Z}) - \left(\nabla^{\mathcal{H}}_{\boldsymbol{X}}\boldsymbol{B}\right)(\boldsymbol{Y},\boldsymbol{Z})\,. \tag{254}$$

Let (X^1, X^2, X^3) be a local coordinate chart for $\mathcal{B}$ such that at any point of the hypersurface $\mathcal{H}$, $\{X^1, X^2\}$ is a local coordinate chart for $\mathcal{H}$ and the normal vector field $\boldsymbol{N}$ to $\mathcal{H}$ is tangent to the coordinate curve X^3 . We say that such a chart is compatible with $\mathcal{H}$. Note that given the symmetries of the curvature tensor and the second fundamental form, the Gauss and Codazzi-Mainardi equations reduce in components to

$$\bar{\mathcal{R}}_{1212} - \mathcal{R}^{\mathcal{H}}_{1212} = B_{11}B_{22} - B_{12}B_{12}\,, \tag{255a}$$

$$\bar{\mathcal{R}}_{1213} = B_{11|2} - B_{21|1}\,, \tag{255b}$$

$$\bar{\mathcal{R}}_{2123} = B_{22|1} - B_{12|2}\,, \tag{255c}$$

where we denote by a stroke $|$ the covariant derivative corresponding to the Levi-Civita connection of $(\mathcal{H}, \boldsymbol{G})$, i.e., $B_{AB|C} = B_{AB,C} - \Gamma^K{}_{CA}B_{KB} - \Gamma^K{}_{CB}B_{AK}$, where $\Gamma^C{}_{AB}$ is the Christoffel symbol of the connection $\nabla^{\mathcal{H}}$ in the local chart $\{X^1, X^2\}$.

The fundamental theorem of surface theory, first proved by [8], implies that the geometry of a surface is fully described by its metric and its second fundamental form [16, 36].

A.3 Geometry of Riemannian Submanifolds

In the following, we tersely review a few elements of the geometry of embedded submanifolds. Here we mainly follow [34, 17, 9, 79] and [42]. Let us consider a Riemannian manifold $\mathcal{S}$ embedded in another Riemannian manifold $\mathcal{Q}$ and assume that $\dim \mathcal{S} < \dim \mathcal{Q}$. We consider a time-dependent embedding $\psi_t : \mathcal{S} \to \mathcal{Q}$. The metric $\boldsymbol{h}$ on $\mathcal{Q}$ induces a metric $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$ on $\mathcal{S}$ (the first fundamental form). At any given point p of $\mathcal{S}$, the tangent space $T_p\mathcal{S}_t$ has an orthogonal complement $(T_p\mathcal{S}_t)^\perp \subset T\mathcal{Q}$ such that

$$T_p\mathcal{Q} = T_p\mathcal{S}_t \oplus (T_p\mathcal{S}_t)^\perp . \tag{256}$$

Note that such a decomposition is smooth in the sense that any smooth vector field $\boldsymbol{u}$ on $\mathcal{S}_t$ can be smoothly decomposed into a vector field $\boldsymbol{u}_\parallel$ tangent to $\mathcal{S}_t$ and a vector field $\boldsymbol{u}_\perp$ normal to $\mathcal{S}_t$, so that $p \to (\boldsymbol{u}_\parallel)_p = (\boldsymbol{u}_p)_\parallel$ and $p \to (\boldsymbol{u}_\perp)_p = (\boldsymbol{u}_p)_\perp$ are smooth. We write $\boldsymbol{u} = \boldsymbol{u}_\parallel + \boldsymbol{u}_\perp$. The orientation of $\boldsymbol{\eta}_i^t$, for $i \in \{1, ..., k\}$, is chosen such that the orientations of $\mathcal{S}_t$ and $\mathcal{Q}$ are consistent in the sense that the orientation induced from $\mathcal{S}_t$ along with the ordered sequence $\{\boldsymbol{\eta}_i^t\}_{i\in\{1,\dots,k\}}$, is equivalent to the orientation of $\mathcal{Q}$. Let $\dim \mathcal{S} = n$ and $\dim \mathcal{Q} = n + k = m$. Following the smoothness of the decomposition (256), one can take a set of smooth vector fields $\{\boldsymbol{\eta}_i^t\}_{i=1,\dots,k}$ normal to $\mathcal{S}_t$ such that they form an orthonormal basis for $\mathfrak{X}^\perp(\mathcal{S}_t)$, the set of vector fields normal to $\mathcal{S}_t$. Let $\{\chi^\alpha\}_{\alpha=1,\dots,n+k}$ be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1, ..., \chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$, and such that the unit normal vector field $\boldsymbol{\eta}_i^t$ for $i \in \{1, ..., k\}$ is tangent to the coordinate curve χ^{n+i}. Hence, every vector field $\boldsymbol{u}$ on $\mathcal{Q}$ along $\mathcal{S}_t$ can be written as $\boldsymbol{u} = \boldsymbol{u}_\parallel + \sum_{i=1}^k u_\perp^i \boldsymbol{\eta}_i^t$.[2] Note that, for $i, j \in \{1, ..., k\}$, one has $\langle\!\langle \boldsymbol{\eta}_i^t, \boldsymbol{\eta}_j^t \rangle\!\rangle_{\boldsymbol{h}} = \delta_{ij}$ and $\langle\!\langle \boldsymbol{\eta}_i^t, \boldsymbol{u}_\parallel \rangle\!\rangle_{\boldsymbol{h}} = 0$, where the Kronecker

[2]In the local coordinate $\{\chi^\alpha\}_{\alpha=1,\dots,n+k}$, we denote $u_\perp^i = u^{n+i}$ for $i \in \{1, ..., k\}$.

delta symbol δ_{ij} is defined as: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Note that at any point of $\mathcal{S}_t$, one has $h_{\alpha(n+i)} = \delta_{\alpha(n+i)}$, for $i \in \{1, ..., k\}$ and $\alpha \in \{1, ..., n+k\}$. We denote the connection coefficients for the Levi-Civita connections $\nabla^{\boldsymbol{h}}$ and $\nabla^{\boldsymbol{g}_t}$ corresponding to the metrics $\boldsymbol{h}$ and $\boldsymbol{g}_t$ by $\tilde{\gamma}^{\alpha}_{\beta\gamma}$ and γ^{a}_{bc} , respectively. We denote by $D^{\boldsymbol{h}}_t$ and $D^{\boldsymbol{g}_t}_t$ the covariant derivatives along $\tilde{\varphi}_X$ and φ_X , respectively. For a vector field $\boldsymbol{u}$ on $\mathcal{Q}$ along $\mathcal{S}_t$, we write $D^{\boldsymbol{h}}_t \boldsymbol{u} = \frac{\partial u^{\alpha}}{\partial t} \tilde{\partial}^t_{\alpha} + \frac{\partial u^i_{\perp}}{\partial t} \boldsymbol{\eta}^t_i + \nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}} \boldsymbol{u}$ and for a vector field $\boldsymbol{w}$ on $\mathcal{S}$, $D^{\boldsymbol{g}_t}_t \boldsymbol{w} = \frac{\partial w^a}{\partial t} \partial^t_a + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}} \boldsymbol{w}$, where $\{\tilde{\partial}^t_{\alpha}\}_{\alpha=1,...,n}$ and $\{\partial^t_a\}_{a=1,...,n}$ denote local coordinate bases for $\mathcal{S}_t$ and $\mathcal{S}$, respectively.

Note that for vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined on $\mathcal{S}_t$ and $\mathcal{Q}$, respectively, such that $\boldsymbol{Y}$ is everywhere tangent to $\mathcal{S}_t$, $\nabla^{\boldsymbol{g}_t}_{\psi^*_t \boldsymbol{X}} \psi^*_t \boldsymbol{Y} = \psi^*_t (\nabla^{\boldsymbol{h}}_{\boldsymbol{X}} \boldsymbol{Y})_{\parallel}$.[3] As a corollary, given a curve c in $\mathcal{S}_t$ and $\boldsymbol{X}$ a vector field along c tangent to $\mathcal{S}_t$ everywhere, $D^{\boldsymbol{g}_t}_s \psi^*_t \boldsymbol{X} = (D^{\boldsymbol{h}}_s \boldsymbol{X})_{\parallel}$, where $D^{\boldsymbol{g}_t}_s = \nabla^{\boldsymbol{g}_t}_{\frac{\partial}{\partial s}}$. For $i \in \{1, ..., k\}$, the i^{th} second fundamental form of $\mathcal{S}_t$ along $\boldsymbol{\eta}^t_i$ is a $\binom{0}{2}$-tensor $\boldsymbol{\kappa}^t_i$ on $\mathcal{S}_t$ defined as [17, 9]

$$\boldsymbol{\kappa}^t_i(\boldsymbol{u}, \boldsymbol{w}) = \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{u}} \boldsymbol{\eta}^t_i, \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad \forall\ \boldsymbol{u}, \boldsymbol{w} \in T_{\chi}\mathcal{S}_t . \tag{257}$$

It is known that $\boldsymbol{\kappa}^t_i$ is a symmetric tensor and can equivalently be written as

$$\boldsymbol{\kappa}^t_i = (\nabla^{\boldsymbol{h}} \boldsymbol{\eta}^t_i)^{\flat}_{\parallel} , \quad i = 1, ..., k .$$

On $\mathcal{S}$, we define, for $i \in \{1, ..., k\}$, the i^{th} second fundamental form as $\boldsymbol{k}^t_i = \psi^*_t \boldsymbol{\kappa}^t_i$. For vector fields $\boldsymbol{u}, \boldsymbol{w} \in T_{\boldsymbol{x}}\mathcal{S}$ one can write $\nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w} = \psi_* \nabla^{\boldsymbol{g}_t}_{\boldsymbol{u}} \boldsymbol{w} + \sum_{i=1}^{k} h^i(\boldsymbol{u}, \boldsymbol{w}) \boldsymbol{\eta}^t_i$, where $h^i(.,.)$ is a bilinear form. Therefore

$$h^i(\boldsymbol{u}, \boldsymbol{w}) = \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w}, \boldsymbol{\eta}^t_i \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad i = 1, ..., k .$$

Knowing that $\langle\!\langle \psi_* \boldsymbol{w}, \boldsymbol{\eta}^t_i \rangle\!\rangle_{\boldsymbol{h}} = 0$ one concludes that

$$\left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w}, \boldsymbol{\eta}^t_i \right\rangle\!\right\rangle_{\boldsymbol{h}} = - \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \boldsymbol{\eta}^t_i, \psi_* \boldsymbol{w}, \right\rangle\!\right\rangle_{\boldsymbol{h}} , \quad i = 1, ..., k .$$

[3] The proof given in [79] still holds even when the embedding is time dependent. Note that $\nabla^{\boldsymbol{g}_t}$ and $\nabla^{\boldsymbol{h}}$ are the Levi-Civita connections corresponding to $\boldsymbol{g}_t$ and $\boldsymbol{h}$, respectively.

Hence

$$h^i(\boldsymbol{u},\boldsymbol{w}) = -\left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_*\boldsymbol{u}}\boldsymbol{\eta}^t_i, \psi_*\boldsymbol{w}, \right\rangle\!\right\rangle_{\boldsymbol{h}} = -(\nabla^{\boldsymbol{h}}\boldsymbol{\eta}^t_i)^\flat(\psi_*\boldsymbol{u},\psi_*\boldsymbol{w}) = -\boldsymbol{k}^t_i(\boldsymbol{u},\boldsymbol{w})\,, \quad i=1,...,k\,.$$

Therefore, we obtain Gauss's equation

$$\nabla^{\boldsymbol{h}}_{\psi_*\boldsymbol{u}}\psi_*\boldsymbol{w} = \psi_*\nabla^{\boldsymbol{g}_t}_{\boldsymbol{u}}\boldsymbol{w} - \sum_{i=1}^k \boldsymbol{k}^t_i(\boldsymbol{u},\boldsymbol{w})\boldsymbol{\eta}^t_i\,.$$

On the other hand, for $i,j\in\{1,...,k\}$, the projection of $\nabla^{\boldsymbol{h}}\boldsymbol{\eta}^t_i$ along $\boldsymbol{\eta}^t_j$ defines $\boldsymbol{\omega}^t_{ij}$, the normal fundamental 1-form of $\mathcal{S}_t$ relative to the unit normals $\boldsymbol{\eta}^t_i$ and $\boldsymbol{\eta}^t_j$. For any vector $\boldsymbol{w}$ tangent to $\mathcal{S}_t$, the 1-form $\boldsymbol{\omega}^t_{ij}$ is defined by [9]

$$\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{w} = \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{w}}\boldsymbol{\eta}^t_i, \boldsymbol{\eta}^t_j\right\rangle\!\right\rangle_{\boldsymbol{h}}\,.$$

Note that, for $i,j\in\{1,...,k\}$, the normal fundamental 1-form $\boldsymbol{\omega}^t_{ij}$ is anti-symmetric, i.e., $\boldsymbol{\omega}^t_{ij} = -\boldsymbol{\omega}^t_{ji}$. On $\mathcal{S}$, one defines the normal fundamental 1-forms, for $i,j\in\{1,...,k\}$, as $\boldsymbol{o}^t_{ij} = \psi^*_t\boldsymbol{\omega}^t_{ij}$. Note that, for a tangent vector field $\boldsymbol{w}$ on $\mathcal{S}_t$, one can write the following[4]

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{w}}\boldsymbol{\eta}^t_i = \boldsymbol{h}^\sharp\cdot\boldsymbol{\kappa}^t_i\cdot\boldsymbol{w} + \sum_{j=1}^k\left(\boldsymbol{\omega}^t_{ij}\cdot\boldsymbol{w}\right)\boldsymbol{\eta}^t_j\,. \tag{258}$$

One needs to be careful in calculating time derivatives in $(\mathcal{S},\boldsymbol{g}_t)$, since the induced metric $\boldsymbol{g}_t$ itself depends on time. In particular, when calculating the derivative of the inner product $\langle\!\langle\boldsymbol{u},\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t}$ of two vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$ along a time-parametrized curve c, the usual formula

$$\frac{d}{dt}\langle\!\langle\boldsymbol{u},\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle D^{\boldsymbol{g}_t}_t\boldsymbol{u},\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle\boldsymbol{u},D^{\boldsymbol{g}_t}_t\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t}\,, \tag{259}$$

is no longer valid when the metric $\boldsymbol{g}_t$ is t-dependent. One instead has[5]

$$\frac{d}{dt}\langle\!\langle\boldsymbol{u},\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle D^{\boldsymbol{g}_t}_t\boldsymbol{u},\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle\boldsymbol{u},D^{\boldsymbol{g}_t}_t\boldsymbol{w}\rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle\boldsymbol{u},\boldsymbol{w}\rangle\!\rangle_{\frac{\partial\boldsymbol{g}_t}{\partial t}}\,, \tag{260}$$

[4]Recall that in the chosen coordinate chart $\{\chi^\alpha\}_{\alpha=1,...,n+k}$, one has $h_{\alpha(n+i)} = \left\langle\!\left\langle\tilde{\partial}^t_\alpha,\boldsymbol{\eta}^t_i\right\rangle\!\right\rangle_{\boldsymbol{h}} = \delta_{\alpha(n+i)}$.

[5]Note that $D^{\boldsymbol{g}_t}_t\boldsymbol{g}_t = \frac{\partial\boldsymbol{g}_t}{\partial t}$.

where

$$\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = u^a v^b \frac{\partial g_{t\,ab}}{\partial t} \,.$$

This can be written in terms of the inner product with respect to $\boldsymbol{g}_t$ as

$$\langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = \left\langle\!\!\left\langle \boldsymbol{u}, \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}}{\partial t} \cdot \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \,,$$

where $\boldsymbol{g}_t^\sharp$ denotes the "inverse metric", with components g_t^{ab}. Therefore[6]

$$\frac{d}{dt} \langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} = \langle\!\langle D_t^{\boldsymbol{g}_t} \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \langle\!\langle \boldsymbol{u}, D_t^{\boldsymbol{g}_t} \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} + \left\langle\!\!\left\langle \boldsymbol{u}, \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \,. \qquad (262)$$

Using the Levi-Civita connection for the metric $\boldsymbol{g}_t$ to calculate covariant derivatives, the symmetry lemma of classical Riemann geometry [44, 56] still holds.[7]

Lemma A.3.1. *For a Riemannian manifold with a time-dependent metric $\boldsymbol{g}_t$*

$$D_\epsilon^{\boldsymbol{g}_t} \frac{\partial c(t,\epsilon)}{\partial t} = D_t^{\boldsymbol{g}_t} \frac{\partial c(t,\epsilon)}{\partial \epsilon} \,.$$

The velocity of the time-dependent embedding ψ_t is defined as

$$\boldsymbol{\zeta} = \frac{\partial \psi(t, \boldsymbol{x})}{\partial t} = \boldsymbol{\zeta}_\parallel + \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{\eta}_i^t \,,$$

where $\boldsymbol{\zeta}_\parallel$ is the tangential velocity of the embedding. We also define $\boldsymbol{Z} := \psi_t^* \boldsymbol{\zeta}_\parallel \circ \varphi_t$.

Lemma A.3.2. *For an arbitrary embedding ψ_t, the following relation holds*

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = \mathfrak{L}_{\boldsymbol{Z}} \boldsymbol{g}_t + 2 \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t \,, \qquad (263)$$

---

[6]It is also possible to define an alternative covariant time derivative, $\tilde{D}_t^{\boldsymbol{g}_t}$, so that an identity analogous to (259) holds. If we let

$$(\tilde{D}_t^{\boldsymbol{g}_t} \boldsymbol{u})^a = \frac{du^a}{dt} + \gamma_{cd}^a u^d \frac{dx^c}{dt} + \frac{1}{2} g^{ab} \frac{\partial g_{bc}}{\partial t} u^c \,, \qquad (261)$$

one readily verifies that

$$\frac{d}{dt} \langle\!\langle \boldsymbol{u}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{g}_t} = \left\langle\!\!\left\langle \tilde{D}_t^{\boldsymbol{g}_t} \boldsymbol{u}, \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} + \left\langle\!\!\left\langle \boldsymbol{u}, \tilde{D}_t^{\boldsymbol{g}_t} \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \,.$$

See [82] for a discussion on this alternative covariant time derivative.

[7]Note that if we were to use the alternative covariant derivative (261), this formula would need to be modified.

where $\mathfrak{L}$ denotes the autonomous Lie derivative.[8] *For a transversal embedding, i.e., when* $\boldsymbol{Z} = \boldsymbol{0}$, (263) *reduces to*

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k} \zeta_{\perp}^{i} \boldsymbol{k}_i^t \,. \tag{264}$$

Proof: First, we note that

$$\boldsymbol{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \left[\frac{d}{dt}\left(\psi_t \circ \psi_s^{-1}\right)^* \boldsymbol{h}\right]_{s=t} = \left[\frac{d}{dt}\psi_{s*}\psi_t^* \boldsymbol{h}\right]_{s=t} = \left[\frac{d}{dt}\psi_{s*}\boldsymbol{g}_t\right]_{s=t} = \psi_{t*}\left[D_t^{\boldsymbol{g}_t}\boldsymbol{g}_t\right]_{s=t} = \psi_{t*}\frac{\partial \boldsymbol{g}_t}{\partial t} \,. \tag{265}$$

On the other hand, we also have

$$\boldsymbol{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \mathfrak{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \mathfrak{L}_{\zeta_{\parallel}}\boldsymbol{h} + \sum_{i=1}^{k} \zeta_{\perp}^{i} \mathfrak{L}_{\boldsymbol{\eta}_i^t}\boldsymbol{h} \,.$$

However, for $i \in \{1, ..., k\}$ one has

$$\left(\mathfrak{L}_{\boldsymbol{\eta}_i^t}\boldsymbol{h}\right)_{\alpha\beta} = (\boldsymbol{\eta}_i^t)_{\alpha|\beta} + (\boldsymbol{\eta}_i^t)_{\beta|\alpha} = 2\kappa_{(i)\alpha\beta} \,. \tag{266}$$

We observe that $\mathfrak{L}_{\zeta_{\parallel}}\boldsymbol{h} = \mathfrak{L}_{\psi_{t*}\boldsymbol{Z}}\psi_{t*}\boldsymbol{g}_t$, and, following [50, p. 98], we have $\mathfrak{L}_{\psi_{t*}\boldsymbol{Z}}\psi_{t*}\boldsymbol{g}_t = \psi_{t*}\mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t$. Thus

$$\boldsymbol{L}_{\boldsymbol{\zeta}}\boldsymbol{h} = \psi_{t*}\left(\mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + 2\sum_{i=1}^{k} \zeta_{\perp}^{i} \boldsymbol{k}_i^t\right) \,. \tag{267}$$

Finally it follows from (265) and (267) that

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = \mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + 2\sum_{i=1}^{k} \zeta_{\perp}^{i} \boldsymbol{k}_i^t \,.$$

□

[8] *The autonomous Lie derivative $\mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t$ is defined by holding the explicit time-dependence of $\boldsymbol{g}_t$ fixed, i.e.,* $\mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t = \frac{d}{ds}\Big|_{t=s}\left[\left(\psi_t \circ \psi_s^{-1}\right)^* \boldsymbol{g}_s\right]$.

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